

Further Mathematics SL

VISIT **Topic**

7

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Geometry

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bscure points to address
may be trivial, and if you

Contents:

- A** Using circle theorems
- B** Concyclic points, cyclic quadrilaterals
- C** Similarity
- D** Intersecting chords theorem
- E** The equation of a circle
- F** Concurrency in a triangle
- G** Further theorems (Apollonius, Stewart, Ptolomy)
- H** Proportionality in right angled triangles
- I** Harmonic ratios
- J** Proportional division
- K** Concurrency and Ceva's theorem
- L** Menelaus' theorem
- M** Euler's line and the 9-point circle



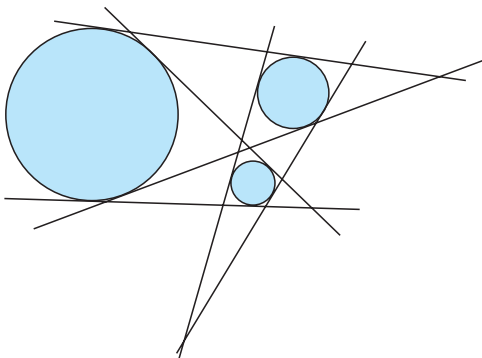
The course itself deals mainly with ratio properties of figures and so we will study in the main, and use:

- Apollonius' theorems
- The theorems of Ceva and Menelaus
- Ptolemy's theorem

Before doing this we need to recall theorems about circles, and congruent and similar figures. Further, since the sine rule is essentially a trigonometric version of similar triangles, it too can be useful in the coming work.

Many amazing discoveries have been made by mathematicians and non-mathematicians who were simply drawing figures with straight edges and compasses.

For example:



This figure consists of three circles of unequal radii. Common external tangents are drawn between each pair of circles and extended until they meet.

Click on the icon to see what interesting fact emerges.



HISTORICAL NOTE



Euclid was one of the great mathematical thinkers of ancient times. It is known that he was the founder of a school in Alexandria during the reign of Ptolemy I, which lasted from 323 BC to 284 BC.

Euclid's most famous mathematical writing is the *Elements*. This work is the most complete study of geometry ever written and has been a major source of information for the study of geometric techniques, logic and reasoning. Despite writing a large number of books on various mathematical topics, Euclid's fame is still for geometry.

A large proportion of the information in the *Elements* was derived from previously written works but the organisation of the material and the discovery of new proofs is credited to Euclid. The importance of his contribution is emphasized by the fact that his *Elements* was used as a text book for 2000 years until the middle of the 19th century when a number of other texts adapting Euclid's original ideas began to appear. After that, the study and teaching of geometry began to follow a variety of paths.

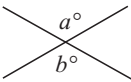
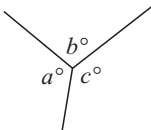
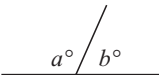
Like many of the great mathematicians and philosophers, Euclid believed in study and learning for its own merit rather than for the rewards it may bring.



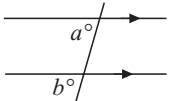
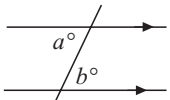
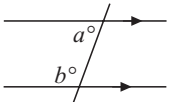
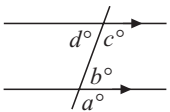
SUMMARY OF PREVIOUSLY PROVEN RESULTS

The following theorems are included for your reference. They provide a reminder of what has been done in previous courses.

ANGLE THEOREMS

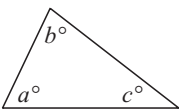
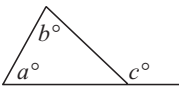
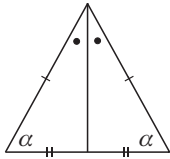
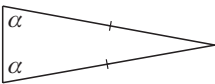
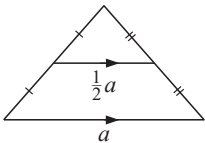
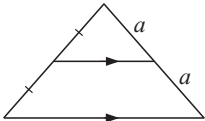
Name	Theorem	Figure
Vertically opposite angles	Vertically opposite angles are equal.	 i.e., $a = b$
Angles at a point	The sum of the angles at a point is 360° .	 i.e., $a + b + c = 360$
Angles on a line	The sum of the angles on a line is 180° .	 i.e., $a + b = 180$

PARALLELISM THEOREMS

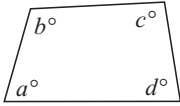
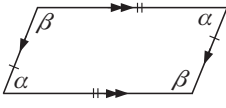
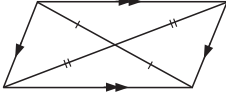
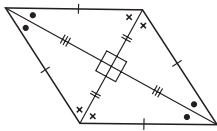
Name	Theorem	Figure
Corresponding angles	When two parallel lines are cut by a third line, then angles in corresponding positions are equal.	 e.g., $a = b$
Alternate angles	When two parallel lines are cut by a third line, then angles in alternate positions are equal.	 e.g., $a = b$
Allied (or co-interior) angles	When two parallel lines are cut by the third line, then angles in allied positions are supplementary.	 e.g., $a + b = 180$
Converse of parallelism theorems	If two lines are cut by a third line, they are parallel if either corresponding angles are equal, alternate angles are equal, or allied angles are supplementary.	e.g.,  l_1 is parallel to l_2 if $a = c$ or $b = d$ or $b + c = 180$.

TRIANGLE THEOREMS

Click on an icon for an interactive demonstration.

Name	Theorem	Figure
Angles of a triangle	The sum of the interior angles of a triangle is 180° .	 i.e., $a + b + c = 180$
Exterior angle of a triangle	The exterior angle of a triangle is equal to the sum of the interior opposite angles.	 i.e., $c = a + b$
Isosceles triangle	In an isosceles triangle: - base angles are equal - the line joining the apex to the midpoint of the base is perpendicular to the base and bisects the angle at the apex.	
Converses of Isosceles triangle theorem	- If a triangle has two equal angles, then the triangle is isosceles. - If the angle bisector of an isosceles triangle bisects the opposite side, it does so at right angles. - If the third angle of a triangle lies on the perpendicular bisector of its base, then the triangle is isosceles.	
The midpoint theorem	The line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.	
Converse to midpoint theorem	The line drawn from the midpoint of one side of a triangle parallel to a second side, bisects the third side.	

QUADRILATERAL THEOREMS

Name	Theorem	Figure
Angles of a quadrilateral	The angles of a quadrilateral add to 360° .	 i.e., $a + b + c + d = 360$
Parallelogram	In a parallelogram: • opposite sides are equal • opposite angles are equal.	
Diagonals of a parallelogram	The diagonals of a parallelogram bisect each other.	
Diagonals of a rhombus	The diagonals of a rhombus • bisect each other at right angles • bisect the angles of the rhombus.	

OTHER IMPORTANT FACTS ABOUT QUADRILATERALS:

Any one of the following facts is sufficient to establish that **a quadrilateral is a parallelogram**:

- opposite sides are equal in length



- one pair of opposite sides is equal and parallel



- opposite angles are equal



- diagonals bisect each other.



Any one of the following facts is sufficient to establish that **a quadrilateral is a rhombus**:

- ▶ the quadrilateral is a parallelogram with one pair of adjacent sides equal
- ▶ the diagonals bisect each other at right angles.

Any one of the following facts is sufficient to prove that **a parallelogram is a rectangle**:

- ▶ one angle is a right angle
- ▶ diagonals are equal in length.

Any one of the following facts is sufficient to establish that **a quadrilateral is a square**:

- ▶ the quadrilateral is a rhombus with one angle a right angle
- ▶ the quadrilateral is a rhombus with equal diagonals
- ▶ the quadrilateral is a rectangle with one pair of adjacent sides equal.

Note:

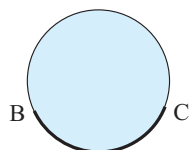
If an arc is less than half the circle it is called a **minor arc**; if it is greater than half the circle it is called a **major arc**.

A chord divides the interior of a circle into two regions called **segments**. The larger region is called a **major segment** and the smaller region is called a **minor segment**.

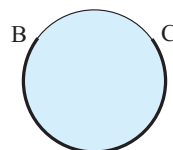
Consider minor arc BC.

We can say that the arc BC **subtends the angle** BAC at A which lies on the circle.

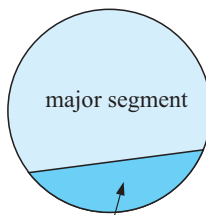
We can also say that the arc BC subtends an angle at the centre of the circle, i.e., angle BOC.



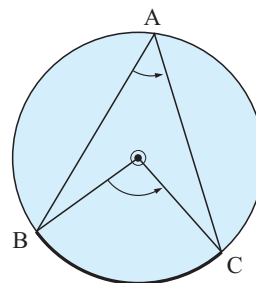
a minor arc BC

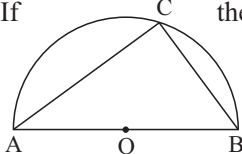
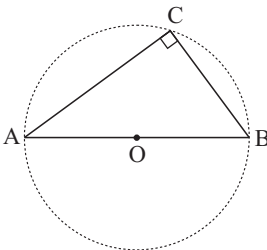
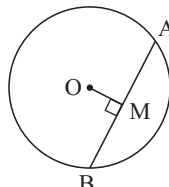
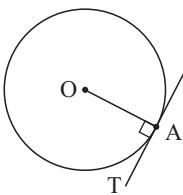


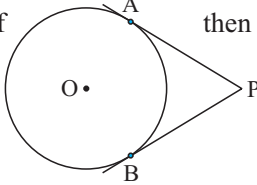
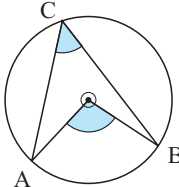
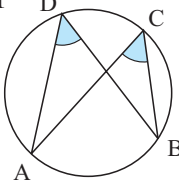
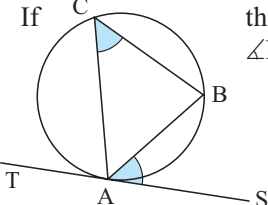
a major arc BC



minor segment

**CIRCLE THEOREMS**

Name of theorem	Statement	Diagram
Angle in a semi-circle	The angle in a semi-circle is a right angle.	If  then $\angle ACB = 90^\circ$. 
Converse of Angle in a semi-circle	If O is the centre of line segment [AB] and [AB] subtends a right angle at C then a circle can be drawn through A, B and C with diameter [AB].	 
Chord of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	If  then $AM = BM$. 
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	If  then $\angle OAT = 90^\circ$. 

Tangents from an external point	Tangents from an external point are equal in length.	If  then $AP = BP$. 
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	If  then $\angle AOB = 2\angle ACB$. 
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	If  then $\angle ADB = \angle ACB$. 
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	If  then $\angle BAS = \angle BCA$. 

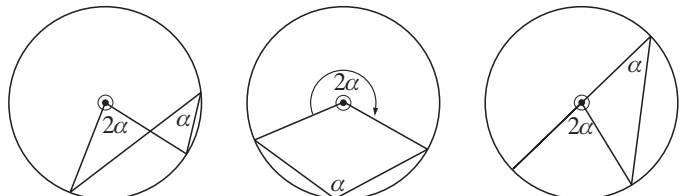
Be prepared to add to this list as the topic progresses.

A useful **converse** is:

The perpendicular bisector of a chord of a circle passes through its centre.

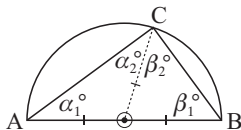
Note:

The following diagrams show other cases of **the angle at the centre theorem**. These cases can be easily shown using the **geometry package**.



SOME PROOFS OF CIRCLE THEOREMS

Although students will not be asked to prove the circle theorems, the following proofs are given to show the rigorous, logical justification needed in proving other results using them.

Angle in a semi-circle

As $OA = OB = OC$, triangles OAC and OBC are isosceles.

$$\therefore \alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2 \quad \{\text{isos. } \Delta \text{ theorem}\}$$

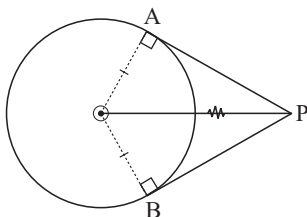
Now in triangle ABC ,

$$\alpha_1 + \beta_1 + (\alpha_2 + \beta_2) = 180 \quad \{\Delta \text{ theorem}\}$$

$$\therefore 2\alpha + 2\beta = 180$$

$$\therefore \alpha + \beta = 90$$

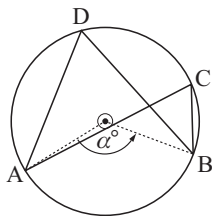
So, $\angle ACB$ is always a right angle.

Tangents from an external point

Triangles OAP and OBP are congruent*,
{RHS} as

- (1) $\angle OAP = \angle OBP = 90$
{tangent-radius theorem}
- (2) $OA = OB$ {equal radii}
- (3) OP is common to both

Consequently, $AP = BP$

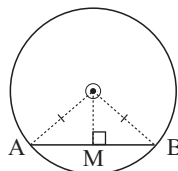
Angles subtended by the same arc

$$\angle ADB = \frac{1}{2}\alpha \quad \{\text{angle at the centre theorem}\}$$

$$\text{and } \angle ACB = \frac{1}{2}\alpha$$

{angle at the centre theorem}

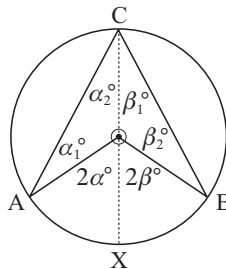
$$\therefore \angle ADB = \angle ACB$$

Chord of a circle

As $OA = OB$ {equal radii}

triangle OAB is isosceles

$$\therefore AM = MB \quad \{\text{isos. } \Delta \text{ theorem}\}$$

Angle at the centre (one case)

As $OA = OC = OB$ {equal radii}

triangles AOC and OBC are isosceles

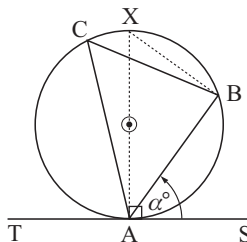
$$\therefore \alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2$$

{isosceles Δ theorem}

But $\angle AOX = 2\alpha$ and $\angle BOX = 2\beta$
{exterior angle of Δ theorem}

$$\therefore \angle AOB = 2\alpha + 2\beta$$

$$= 2 \times \angle ACB$$

Angle between a tangent and a chord

We draw AOX and BX .

$$\angle XAS = 90 \quad \{\text{tangent-radius}\}$$

$$\angle ABX = 90 \quad \{\text{angle in semi-circle}\}$$

$$\text{Let } \angle BAS = \alpha, \angle BAX = 90 - \alpha$$

So, in ΔABX ,

$$\angle BXA = 180 - 90 - (90 - \alpha) = \alpha$$

But $\angle BXA = \angle BCA$

{angles in same segment}

$$\therefore \angle BCA = \angle BAS = \alpha$$

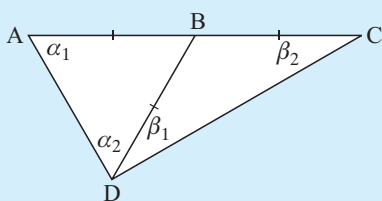
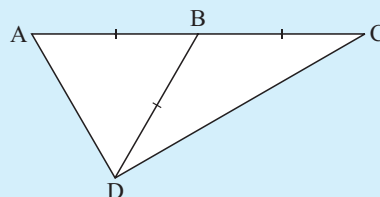
* Congruence of triangles is revised later in the chapter

A

USING CIRCLE THEOREMS

Example 1

Show that angle ADC is a right angle:



Triangle ABD is isosceles as

$$AB = BD \quad \{\text{given}\}$$

$$\therefore \alpha_1 = \alpha_2 \quad \{\text{isosceles triangle theorem}\}$$

Likewise, $\beta_1 = \beta_2$ in isosceles triangle BCD.

Thus in triangle ADC

$$\alpha + (\alpha + \beta) + \beta = 180$$

{angles of a triangle theorem}

$$\therefore 2\alpha + 2\beta = 180$$

$$\therefore \alpha + \beta = 90$$

$\therefore \angle ADC$ is a right angle.

Alternatively:

Since $BA = BC = BD$, a circle with centre B can be drawn through A, D and C with [AC] being a diameter.

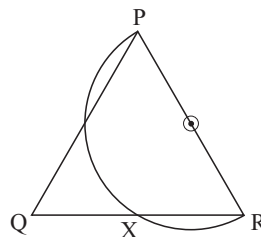
Thus $\angle ADC$ is a right angle.

{angle in a semi-circle theorem}

EXERCISE A

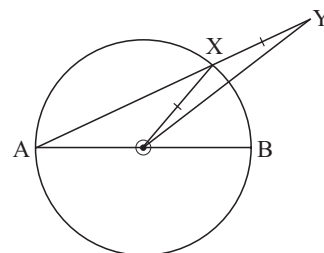
- 1 Triangle PQR is isosceles with $PQ = PR$. A semi-circle with diameter [PR] is drawn and it cuts [QR] at X.

Prove that X is the midpoint of [QR].



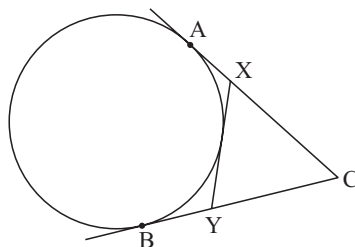
- 2 [AB] is the diameter of a circle centre O. X is a point on the circle and [AX] is produced to Y such that $OX = XY$.

Prove that angle YOY is three times the size of angle XOY.



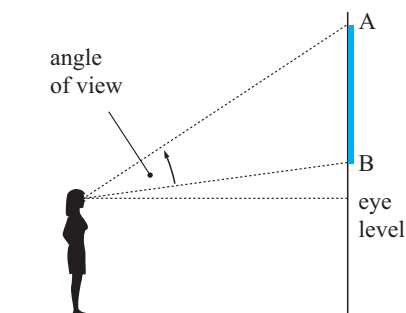
- 3 Triangle PQR is isosceles with $PQ = QR$. PQR is inscribed in a circle (its vertices lie on the circle). [XP] is a tangent to the circle. Prove that [QP] bisects angle XPR.

- 4** [AB] is a diameter of a circle, centre O. [CD] is a chord parallel to [AB]. Prove that [BC] bisects the angle DCO, regardless of where [CD] is located.
- 5** [PQ] and [RS] are two perpendicular chords of a circle, centre O. Prove that $\angle POS$ and $\angle QOR$ are supplementary.
- 6** The bisector of $\angle YXZ$ of $\triangle XYZ$ meets [YZ] at W. When a circle is drawn through X it touches [YZ] at W and cuts [XY] and [XZ] at P and Q respectively. Prove that $\angle YWP = \angle ZWQ$.
- 7** A, B and C are three points on a circle. The bisector of angle CAB cuts [BC] at P and the circle at Q. Prove that $\angle APC = \angle ABQ$.
- 8** [AB] and [DC] are parallel chords of a circle. [AC] and [BD] intersect at E. Prove that:
- a** triangles ABE and CDE are isosceles **b** $AC = BD$.
- 9** P is any point on the circle. [QR] is a chord of the circle parallel to the tangent at P. Prove that triangle PQR is isosceles.
- 10** Triangle ABC is inscribed in a circle and $AB = AC$. The bisector of angle ACB meets the tangent from A at D. Prove that [AD] and [BC] are parallel.
- 11** Triangle PQR is inscribed in a circle with [PR] as a diameter. The perpendicular from P to the tangent at Q meets the tangent at S. Prove that [PQ] bisects angle SPR.
- 12** Tangents are drawn from fixed point C to a fixed circle, meeting it at A and B. [XY] is a moving tangent which meets [AC] at X and [BC] at Y. Prove that triangle XYZ has constant perimeter.

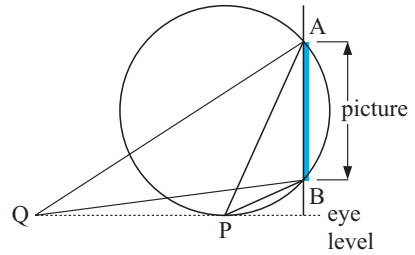


- 13** Two circles touch internally at point P. The tangent to the inner circle at Q meets the outer circle at R and S. Prove that [QP] bisects angle RPS.
- 14** [AB] is a diameter of a circle. The tangent at X cuts the diameter produced at Y. [XZ] is perpendicular to [AY]. Prove that [XA] and [XB] are the internal and external bisectors of $\angle ZXY$.

- 15** Britney notices that her angle of view of a picture on a wall depends on how far she is standing in front of the wall. When she is close to the wall the angle of view is small. When she moves backwards so that she is a long way from the wall the angle of view is also small. It becomes clear to Britney that there must be a point in the room where the angle of view is greatest. She is wondering whether this position can be found from a deductive geometry argument only. Kelly said that she thought this could be done by drawing an appropriate circle.

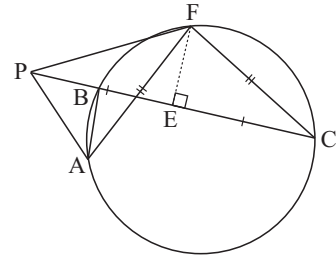


She said that the solution is to draw a circle through A and B which touches the 'eye level' line at P, then $\angle APB$ is the largest angle of view. To prove this, choose any other point Q on the eye level line and show that this angle must be less than $\angle APB$. Complete the full argument.



- 16** In the given figure $AF = FC$ and $PE = EC$.

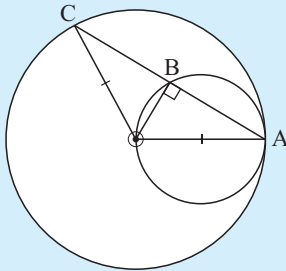
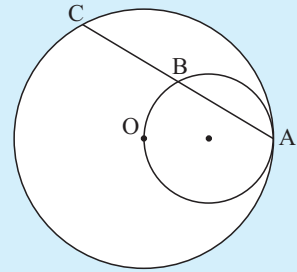
- Prove that triangle FPA is isosceles.
- Prove that $AB + BE = EC$.



Example 2

Given a circle, centre O, and a point A on the circle, a smaller circle of diameter [OA] is drawn. [AC] is any line drawn from A to the larger circle, cutting the smaller circle at B.

Prove that the smaller circle will always bisect [AC].



Join [OA], [OC] and [OB].

Now $\angle OBA$ is a right angle.

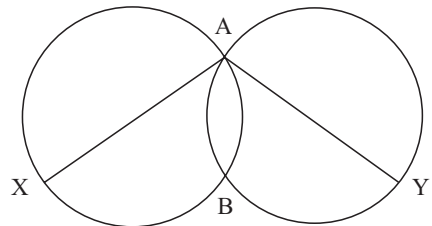
{angle in a semi-circle theorem}

Thus [OB] is the perpendicular from the centre of the circle to the chord [AC].

$\therefore$ [OB] bisects [AC]. {chord of circle theorem}

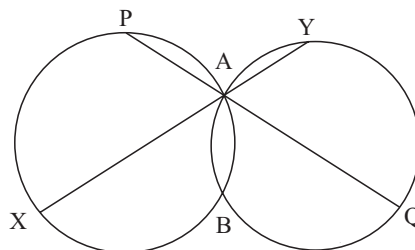
Thus B always bisects [AC].

- 17** Two circles intersect at A and B. [AX] and [AY] are diameters, as shown. Prove that X, B and Y are collinear.

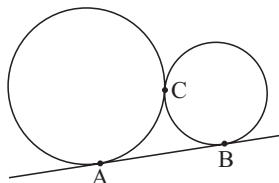


- 18** Two circles intersect at A and B. Straight lines [PQ] and [XY] are drawn through A to meet the circles as shown.

Show that $\angle XBP = \angle YBQ$.



19



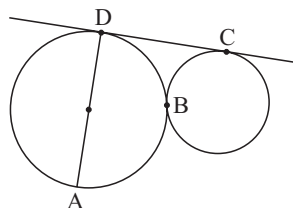
AB is a common tangent to two circles. Prove that:

- the tangent through the point of contact C bisects [AB]
- $\angle ACB$ is a right angle.

- 20** Two circles touch externally at B and [CD] is a common tangent touching the circles at D and C.

[DA] is a diameter.

Prove that A, B and C are collinear.



B

CONCYCLIC POINTS, CYCLIC QUADRILATERALS

A circle can always be drawn through any three non-collinear points.

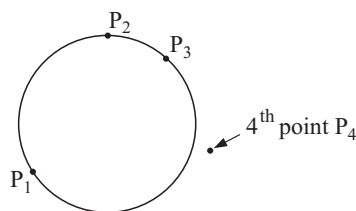
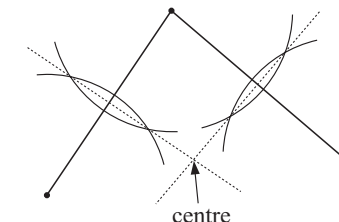
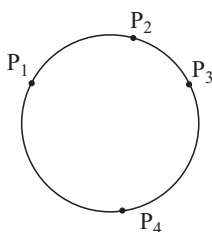
To find the circle's centre we draw the perpendicular bisector of the line joining two pairs of points.

The centre is at the intersection of these two lines.

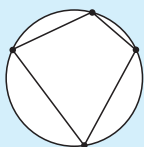
What theorem or converse enables us to do this?

Notice that a circle may or may not be drawn through any four points in the plane.

For example:



If a circle can be drawn through four points we say that the **points are concyclic**.

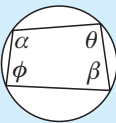


If any four points on a circle are joined to form a convex quadrilateral then the quadrilateral is said to be a **cyclic quadrilateral**.



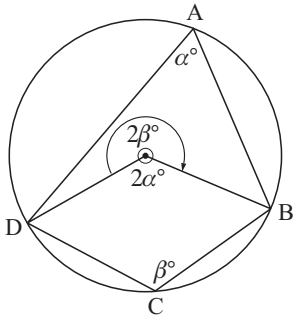
OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL THEOREM

The opposite angles of a cyclic quadrilateral are supplementary,

i.e., given  then $\alpha + \beta = 180^\circ$ and $\theta + \phi = 180^\circ$.



Proof:



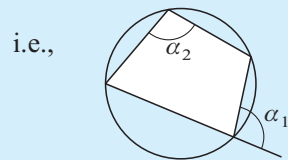
Consider a cyclic quadrilateral ABCD in a circle centre O. Join [OD] and [OB].

If $\angle DAB = \alpha$ and $\angle DCB = \beta$ then $\angle DOB = 2\alpha$
and reflex $\angle DOB = 2\beta$ {angle at the centre theorem}

But $2\alpha + 2\beta = 360$ {angles at a point theorem}
 $\therefore \alpha + \beta = 180$

i.e., angles DAB and DCB are supplementary.
Similarly angles ADC and ABC are supplementary.

Theorem: The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



$$\alpha_1 = \alpha_2.$$

The proof is left to the reader.

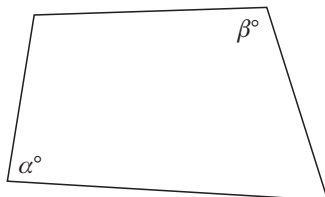


TESTS FOR CYCLIC QUADRILATERALS

A quadrilateral is a **cyclic quadrilateral** if:

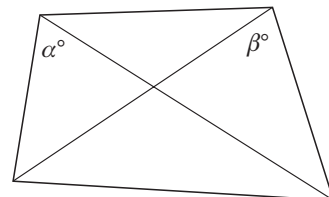
- one pair of opposite angles are supplementary *or*
- one side subtends equal angles at the other two vertices,

i.e.,



where $\alpha + \beta = 180$

or



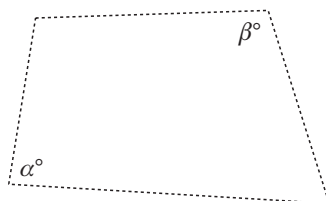
where $\alpha = \beta$

TEST FOR CONCYCLIC POINTS

Four points are **concylic**:

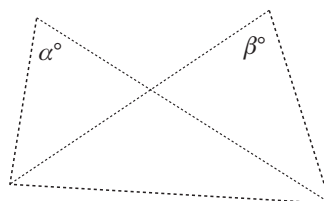
- when the points are joined to form a convex quadrilateral and one pair of opposite angles are supplementary *or*
- when two points (defining a line) subtend equal angles at the other two points on the same side of the line,

i.e.,



where $\alpha + \beta = 180$

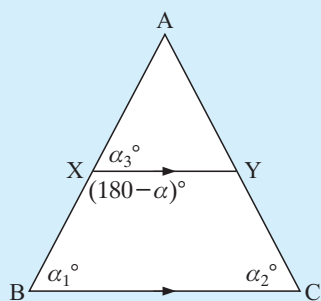
or



where $\alpha = \beta$

Example 3

Triangle ABC is isosceles with $AB = AC$. X and Y lie on [AB] and [AC] respectively such that [XY] is parallel to [BC]. Prove that XYCB is a cyclic quadrilateral.



Since $\triangle ABC$ is isosceles with $AB = AC$, then

$$\alpha_1 = \alpha_2 \quad \{\text{equal base angles}\}$$

Now $XY \parallel BC \Rightarrow \alpha_1 = \alpha_3$
 $\{\text{equal corresponding angles}\}$

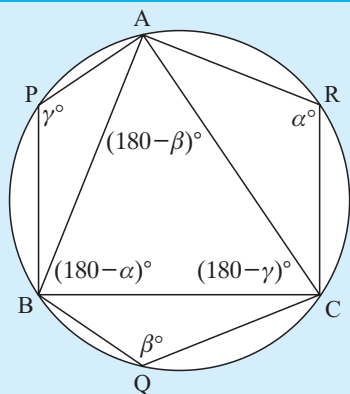
$$\therefore \angle YXB = 180 - \alpha$$

$$\text{Thus } \angle YXB + \angle YCB = 180 - \alpha + \alpha = 180$$

$\therefore$ XYCB is a cyclic quadrilateral
 $\{\text{opposite angles supplementary}\}$

Example 4

Triangle ABC is inscribed in a circle. P, Q and R are any points on arcs AB, BC and AC respectively. Prove that angles ARC, CQB and BPA have a sum of 360° .



Let angles ARC, CQB and BPA be α , β and γ respectively. Now ARCB is a cyclic quadrilateral.

$$\therefore \angle ABC = 180 - \alpha$$

Likewise in cyclic quadrilaterals ABQC and CAPB,

$$\angle BAC = 180 - \beta \quad \text{and} \quad \angle ACB = 180 - \gamma$$

$$\text{Thus } (180 - \alpha) + (180 - \beta) + (180 - \gamma) = 180$$

$\{\text{angle sum of triangle}\}$

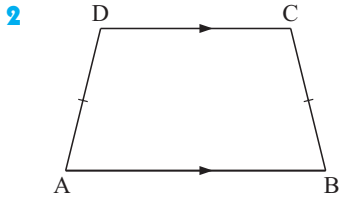
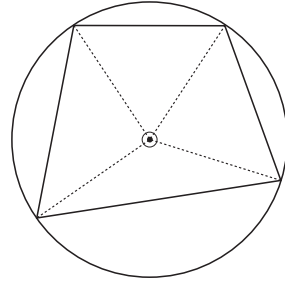
$$\therefore 540 - (\alpha + \beta + \gamma) = 180$$

$$\therefore 360 = \alpha + \beta + \gamma$$

Thus proving the statement.

EXERCISE B

- 1 Show how to use the given figure to prove that the opposite angles of a cyclic quadrilateral are supplementary.

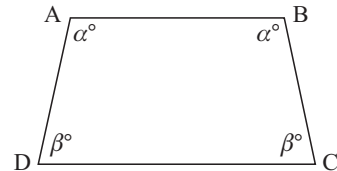


Without assuming any properties of isosceles trapezia, prove that an isosceles trapezium is always a cyclic quadrilateral.

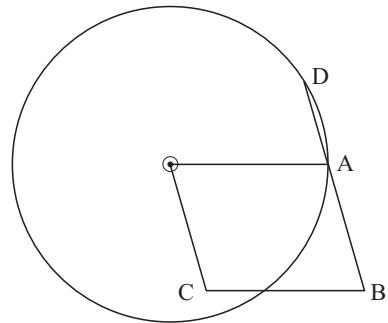
(Hint: Draw [CX] parallel to [DA] meeting [AB] at X.)

- 3 What can be deduced about the quadrilateral ABCD?

Give a detailed argument, with reasons.



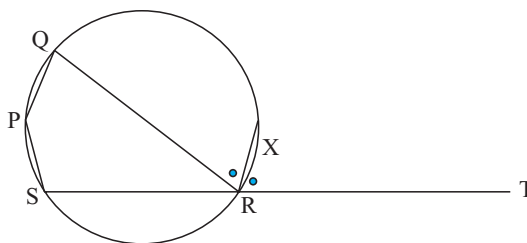
- 4 ABC is an isosceles triangle in which $AB = AC$. The angle bisectors at B and C meet the sides [AC] and [AB] at X and Y respectively. Show that BCXY is a cyclic quadrilateral.
- 5 Two circles meet at points X and Y. [AXB] and [CYD] are two line segments which meet one circle at A and C and the other at B and D. Prove that [AC] is parallel to [BD].
- 6 Prove that a parallelogram inscribed in a circle is a rectangle.
- 7 ABCD is a cyclic quadrilateral and X is any point on diagonal [CA]. [XY] is drawn parallel to [CB] to meet [AB] at Y and [XZ] is drawn parallel to [CD] to meet [AD] at Z. Prove that XYAZ is a cyclic quadrilateral.
- 8 OABC is a parallelogram.
A circle, centre at O and radius [OA] is drawn.
[BA] produced meets the circle at D.
Prove that DOCB is a cyclic quadrilateral.



- 9 Two circles intersect at X and Y. A line segment [AXB] is drawn cutting the circles at A and B respectively. The tangents at A and B meet at C. Prove that AYBC is a cyclic quadrilateral.

- 10** [RX] is the bisector of angle QRT.

Prove that [PX] bisects angle QPS.

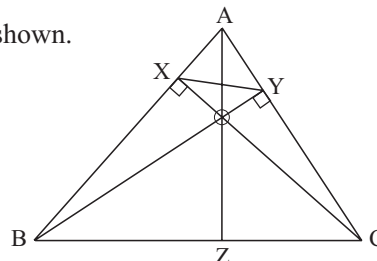


- 11** [AB] and [CD] are two parallel chords of a circle, centre O. [AD] and [BC] meet at E. Prove that A, E, O and C are concyclic points.

- 12** [AB] and [AC] are chords of a circle, centre O. X and Y are the midpoints of [AB] and [AC] respectively. Prove that O, X, A and Y are concyclic points.

- 13** Triangle ABC has perpendiculars [CX] and [BY] as shown.

- What can be said about quadrilaterals AXOY and BXYC? Give reasons.
- Prove that $\angle XAO = \angle XYO = \angle XCB$.
- Prove that [AZ] is perpendicular to [BC].



- 14** Two circles intersect at P and Q. [APB] and [CQD] are two parallel lines which meet the circles at A, B, C and D. Prove that $AB = CD$.

- 15** In triangle PQR, $PQ = PR$. If S and T are the midpoints of [PQ] and [PR] respectively, show that S, Q, R and T are concyclic points.

- 16** Triangle ABC is acute angled and squares ABDE and BCFG are drawn externally to the triangle. If [GA] and [CD] meet at P, show that:

- B, G, C and P are concyclic
- [DC] and [AG] are perpendicular
- [BP] bisects angle DPG.

- 17** [AOB] is a diameter of a circle, centre O. C is any other point on the circle and the tangents at B and C meet at D. Prove that [OD] and [AC] are parallel.

- 18** [AOB] is a diameter of a circle, centre O. C and D are points on the circle such that [AC] bisects $\angle BAD$. The tangent drawn at C cuts [AD] at E. Show that $\angle CEA$ is a right angle.

- 19** Triangle PQR is inscribed in a circle. [ST] is parallel to the tangent at P, intersecting [PQ] at S and [PR] at T. Prove that SQRT is a cyclic quadrilateral.

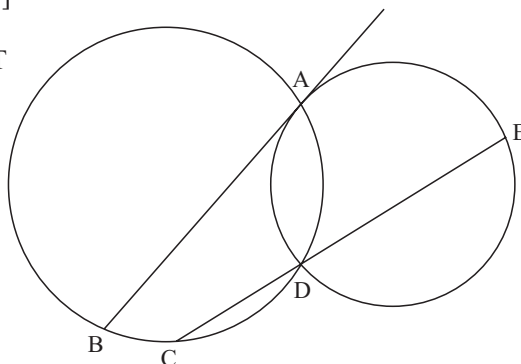
- 20** Two circles meet at A and D.

The tangent at A for one of the circles meets the other circle at B.

A point C is chosen on minor arc BD.

[CD] is produced to E.

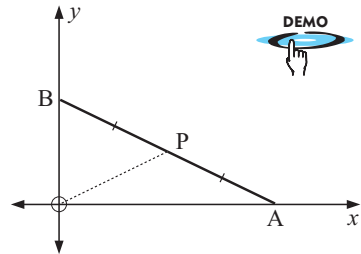
Show that [AE] is parallel to [BC].



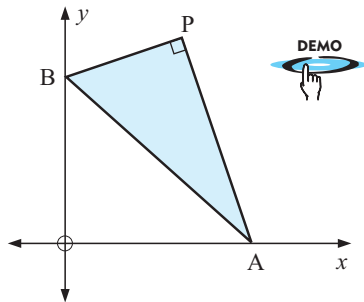
- 21** Two circles intersect at P and Q. [QP] is produced to R, [RS] is a tangent to one circle and [RT] is a tangent to the other. Prove that $RS = RT$.
- 22** [POQ] is a diameter of a circle, centre O, and S is any other point on the circle. [PT] is perpendicular to the tangent at S. Show that [PS] bisects angle TPQ.
- 23** [POQ] is a diameter of a circle, centre O, and R is any other point on the circle. The tangent at R meets the tangents at P and Q at S and T respectively. Show that $\angle SOT$ is a right angle.
- 24** [PQ] and [PR] are tangents to a circle, centre O, from an external point P. [PS] is perpendicular to [PQ] and meets [OR] produced at S. [QR] produced meets [PS] produced at T. Show that $\triangle STR$ is isosceles.
- 25** A solid bar AB moves so that A remains on the x -axis and B remains on the y -axis. At P, the midpoint of AB, is a small light.

Prove that as A and B move to all possible positions, the light traces out a path which forms a circle.

[Do not use coordinate geometry methods.]



26



PAB is a wooden set square in which $\angle APB$ is a right angle. The set square is free to move so that A is always on the x -axis and B is always on the y -axis.

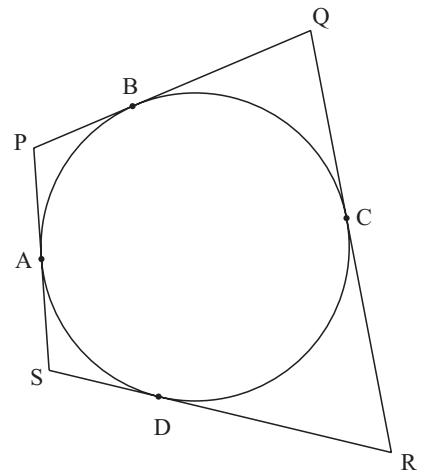
Show that the point P always lies on a straight line segment which passes through O.

[Do not use coordinate geometry methods.]

- 27** Tangents P, Q, R and S form a quadrilateral. This is called a **circumscribed polygon**.

What can be deduced about the opposite sides of the circumscribed quadrilateral?

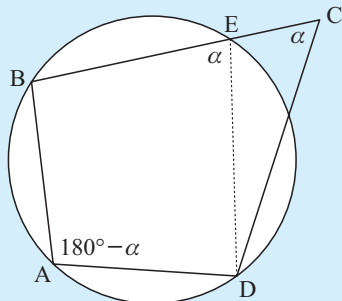
Prove your conjecture.



PROOF OF SOME CONVERSES

Example 5

Prove the converse of the ‘opposite angles of a cyclic quadrilateral theorem’, i.e., if a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral.



Let ABCD be a quadrilateral with $\angle BCD = \alpha$ and $\angle BAD = 180^\circ - \alpha$.

We now draw a circle through A, B and D.

This circle cuts [BC] or [BC] produced in E, as shown. Now join DE.

Clearly, ABED is a cyclic quadrilateral.

Consequently, $\angle BED = \alpha$ {cyclic quadrilateral theorem}

Now $\angle BED = \angle BCD = \alpha$

$\Rightarrow [ED] \parallel [CD]$ {equal corresponding angles}

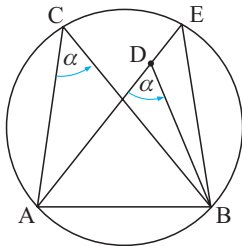
$\Rightarrow$ E and C coincide

$\Rightarrow$ ABCD is a cyclic quadrilateral.

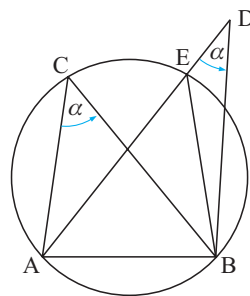
28 Repeat the proof of **Example 5** but this time with C appearing inside the circle.

29 Prove that ‘if a line segment [AB] subtends equal angles at C and D then A, B, C and D are concyclic’. Consider both cases:

a

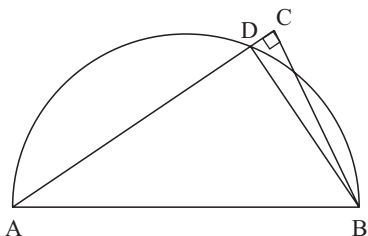


b

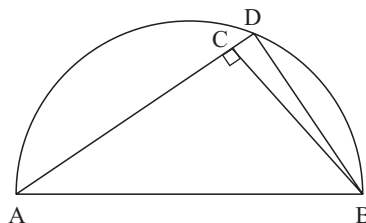


30 Prove that ‘if a line joining two points A and B, say, subtends a right angle at a third point, then this point will lie on a circle with diameter [AB]’. Consider both cases:

a



b



C

SIMILARITY

CONGRUENCE

Two figures are **congruent** if they are exactly the same *shape* (and thus corresponding angles are equal) **and** the same *size* (the ratio of the lengths of corresponding sides is equal to one). Both conditions are required for congruence.

In the case of triangles, these results can be reduced to the following four tests for congruence:

- SSS, where the three corresponding sides are shown to be equal.
- SAS, where two corresponding sides and the *included* angle are shown to be equal.
- ASA, where two corresponding angles and any corresponding side are shown to be equal.
- RHS, where, in a right angled triangle, the two hypotenuses and a pair of corresponding sides are shown to be equal.

Each of the above tests is equivalent to the triangles being proven to be congruent.

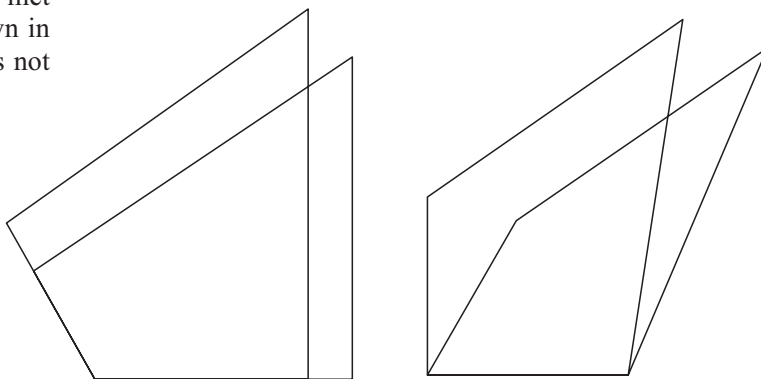
There is a further **ambiguous** case (ASS) where the angle is **not** included between the sides. In this case the two triangles may or may not be congruent (they are related closely if they are not congruent) and the result is equivalent to the ambiguous case of the sine rule.

SIMILARITY

Two figures are **similar** if they are exactly the same *shape* (and thus corresponding angles are equal in size) but can be obtained from each other by an *enlargement*. This implies that the ratio of the lengths of corresponding sides is the constant.

Both conditions must be met for similarity to be shown in most figures, one only is not sufficient.

See opposite:



In the first diagram, the corresponding angles are equal but the corresponding ratios of the sides are not. In the second, the opposite is the case.

Triangles, however, are unique in that they are always similar whenever their angles are the same size. Then, the ratios of the corresponding sides in each triangle are the same (and vice versa).

Usually equal angles are easiest to see, and from this we can make deductions about ratios of lengths of line segments. The connection between angles and lengths is very much the focus of the course and so similarity plays a major role. You would be advised to remember this.

Definition: Two triangles are **similar** if one is an enlargement of the other. Consequently, similar triangles are **equiangular**. Similar triangles have corresponding sides in the same **ratio**.

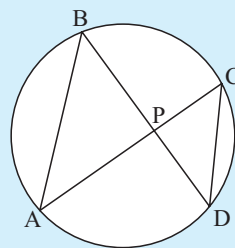
NECESSARY AND SUFFICIENT CONDITIONS FOR SIMILAR TRIANGLES

- ▶ If two triangles are similar then:
 - the triangles are equiangular
 - the corresponding sides are in the same ratio.
- ▶ A pair of triangles is similar if any one of the following is true:
 - the triangles are equiangular
 - the corresponding sides of the triangle are in the same ratio
 - two sides of each triangle are in the same ratio and the included angles are equal.

It is worth stating that congruence and similarity are examples of equivalence relations, a subject which is covered in detail in the Group Theory and Abstract Algebra option. They satisfy the three properties of **reflectivity**, **symmetry** and **transitivity**.

Example 6

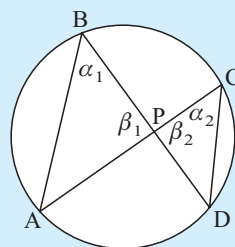
- a Show that the figure contains similar triangles.
- b List the vertices in corresponding order.
- c What is the equation of corresponding side ratios?



- a $\alpha_1 = \alpha_2$ {angles in same segment}
 $\beta_1 = \beta_2$ {vertically opposite}
 This is sufficient to show that the triangles are equiangular and so are similar.
 {The third pair must also be equal.}

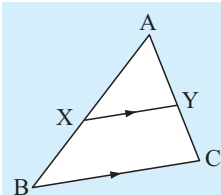
- b $\therefore \triangle s \text{ ABP and DCP are similar.}$

- c Consequently $\frac{AB}{DC} = \frac{BP}{CP} = \frac{AP}{DP}$



Note: $\triangle s \text{ ABP and DCP are similar}$ is sometimes written as $\frac{\triangle ABP}{\triangle DCP}$ are similar.

Theorem:



If $[XY]$ is parallel to $[BC]$ then

$$\frac{AX}{XB} = \frac{AY}{YC}.$$

Proof:

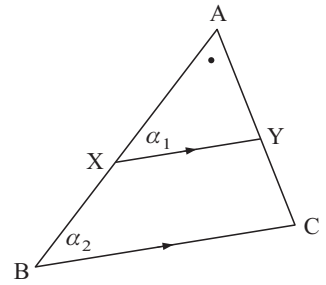
Δ s AXY and ABC are equiangular and so they are similar.

$$\therefore \frac{AX}{AY} = \frac{AB}{AC}$$

$$\therefore \frac{AX}{AY} = \frac{AX + BX}{AY + YC}$$

$$\therefore \cancel{AX \cdot AY} + AX \cdot YC = \cancel{AX \cdot AY} + AY \cdot BX$$

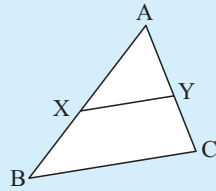
$$\therefore \frac{AX}{BX} = \frac{AY}{YC} \quad (\text{QED})$$



Notation: We write $AB \times CD$ as $AB \cdot CD$

The **converse** is also true,

i.e.,



If $\frac{AX}{AY} = \frac{BX}{CY}$, then $[XY] \parallel [BC]$.

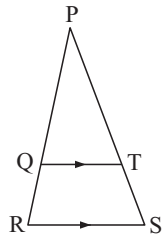
The proof is left to the reader.

EXERCISE C

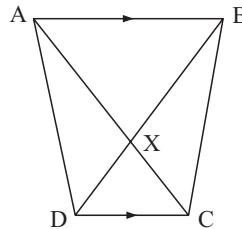
1 For the following figures:

- i identify any similar triangles and prove that they are similar
- ii write an equation connecting lengths of corresponding sides.

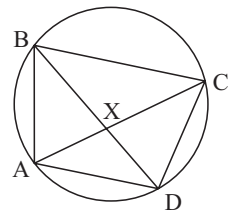
a



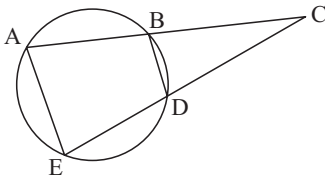
b



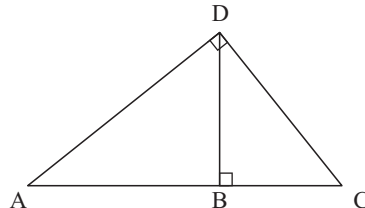
c



d



e

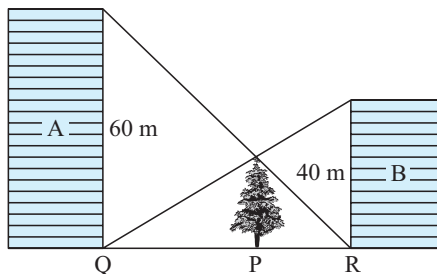


- 2 The tangent at P to a circle meets the chord [QR] produced at the point S. Prove that triangles SPQ and SPR are similar.

- 3 ABCD is a trapezium with [AB] parallel to [DC]. The diagonals of the trapezium meet at M. Prove that $\triangle ABM$ is similar to $\triangle CDM$.

- 4 A pine tree grows between two buildings A and B. On one day it was observed that the top of A, the apex of the tree, and the foot of B line up and at the same time, the foot of A, the apex and the top of B line up, as illustrated. Find the height of the tree.

[Hint: Let the tree's height be h m and $QP = a$ m, $PR = b$ m.]



- 5 PQRS is a cyclic quadrilateral with diagonals meeting at A. Prove that $\triangle PQA$ is similar to $\triangle RSA$.

- 6 PQRS is a parallelogram and T lies on [PS]. [QT] produced meets [RS] produced at U.

Prove that $QT \cdot PS = QU \cdot PT$. (Hint: Prove $\frac{QT}{QU} = \frac{PT}{PS}$ first.)

- 7 ABC is an isosceles triangle with $AB = AC$. X lies on [AC] such that $CB^2 = CX \cdot CA$. Prove that $BX = BC$.

- 8 Triangle ABC has P the midpoint of [BC] and Q the midpoint of [AC]. Medians [AP] and [BQ] are drawn. (A median of a triangle is a line segment from a vertex to the midpoint of the opposite side.). The medians meet at G.

a Prove that $\triangle ABG$ is similar to triangle PQG.

b Hence prove that $GP = \frac{1}{3}AP$.

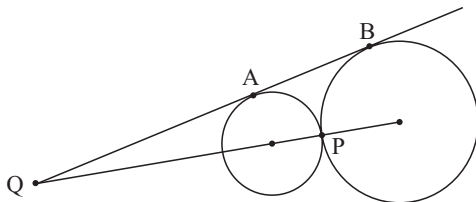
c Explain why all three medians of a triangle meet at the one point.

- 9 Triangle PQR is inscribed in a circle. The angle bisector of $\angle QPR$ meets [QR] at S and the circle at T.

Prove that $PQ \cdot PR = PS \cdot PT$.

- 10 Triangle ABC has altitudes [AP] and [BQ] where P lies on [BC] and Q lies on [AC]. Prove that $AH \cdot HP = BH \cdot HQ$.

11



In the given figure, prove that $QP^2 = QA \cdot QB$.

- 12 [PQ] is a chord of a circle. R lies on the major arc of the circle. Tangents are drawn through P and through Q. From R, perpendiculars [PA], [PB] and [PC] are drawn to the tangent at A, the tangent at B and [AB] respectively. Prove that $RA \cdot RB = RC^2$.

D

INTERSECTING CHORDS THEOREM

INVESTIGATION 1

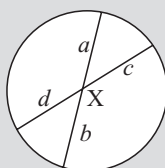
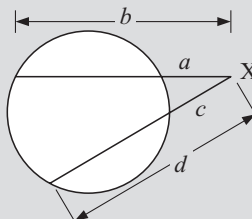
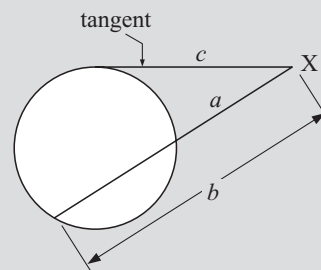
INTERSECTING CHORDS



Click on the icon to access intersecting chords software.

What to do:

1 For the cases:

a**b****c**

Use the software to find the connection between the variables.

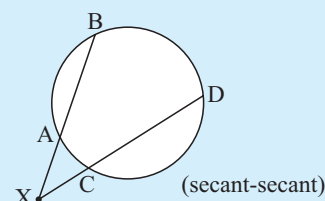
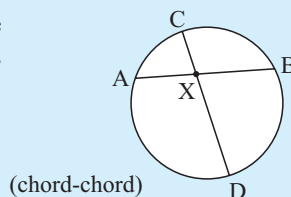
2 Prove that each result is valid by using similar triangles.

In the investigation you should have discovered the **intersecting chords theorem**.

INTERSECTING CHORDS THEOREM

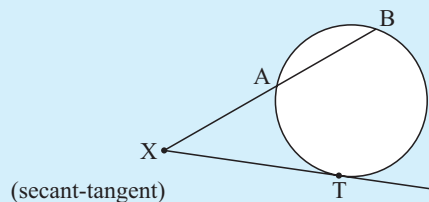
Whether X lies inside or outside a circle with intersecting chords [AB] and [CD], the result

$AX \cdot BX = CX \cdot DX$ holds.

**Special case**

If the tangent at T to a circle meets the chord [BA] produced at X then

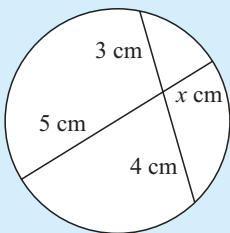
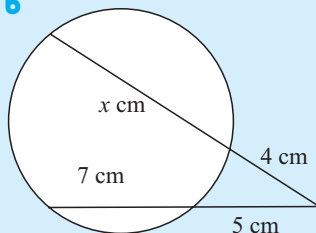
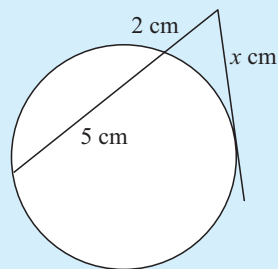
$$XA \cdot XB = XT^2$$



The converses of these theorems also hold.

For example:

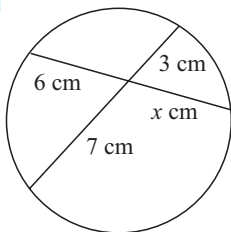
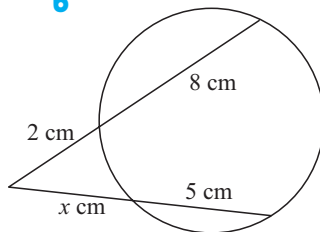
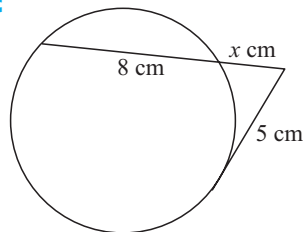
If [AB] and [CD] meet at X and $AX \cdot BX = CX \cdot DX$ then ABCD is a cyclic quadrilateral.

Example 7Find x in:**a****b****c**

a By the intersecting chords theorem, $x \times 5 = 3 \times 4$
 $\therefore 5x = 12$
 $\therefore x = 2.4$

b By the intersecting chords theorem, $4(4 + x) = 5 \times (5 + 7)$
 $\therefore 4(4 + x) = 5 \times 12$
 $\therefore 4 + x = 15$
 $\therefore x = 11$

c $x^2 = 2 \times 7$ $\therefore x^2 = 14$ {special case}
 $\therefore x = \sqrt{14}$ {as $x > 0$ }

EXERCISE D**1** Find x in:**a****b****c****2** Chords [AB] and [CD] meet at X inside the circle.

- a** If $AX = 4$ cm, $BX = 6$ cm and $CX = 5$ cm, find the length of [DX].
- b** If $AX = 2$ cm, $AB = 8$ cm and $CX = 3$ cm, find the length of [CD].
- c** If $AX = 3$ cm, $BX = 5$ cm and $CD = 9$ cm, find the length of [CX].
- d** If $AX = 3$ cm, $BX = 5$ cm and $OX = 4$ cm where O is the circle's centre, find the radius of the circle.
- e** If $BX = 2AX$, $DX = 3$ cm and $CD = 7$ cm, how long is [AB]?

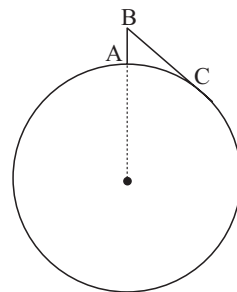
3 Chords [AB] and [CD] of a circle are produced to X where X is outside the circle.

- a** If $BX = 4$ cm, $BA = 2$ cm and $DX = 3$ cm, find the length of [CD].
- b** If $AX = 3BX$, $DX = 3$ cm and $CX = 11$ cm, find the length of [AB].

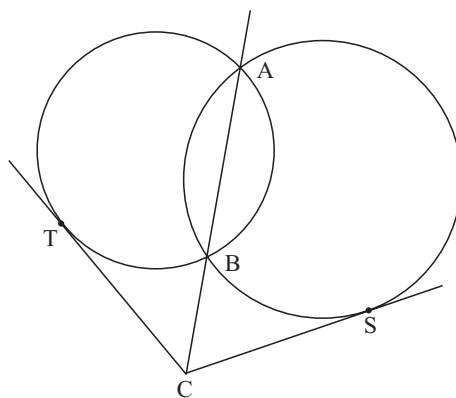
- 4 From X outside a circle centre O, XAB is drawn cutting the circle at A and B. [XT] is a tangent with T the point of contact.
- If $XT = 6$ cm and $XA = 4$ cm, find the length of [BX].
 - If $XA = 2$ cm and $AB = 3$ cm, find the length of [XT].
 - If $XA = 8$ cm, $AB = 2$ cm and $OA = 5$ cm, find the length of [OX].

- 5 The distance of the visible horizon from a point B, above the surface of the earth at A, is the length of the tangent [BC].

- If the radius of the earth is 6370 km, find the distance of the visible horizon from the observers in a space shuttle 400 km above the earth's surface.
- Show that for a height h metres above the earth's surface the visible horizon is given by $D \approx 3.57\sqrt{h}$ km.



- 6 Two circles intersect at A and B. C is any point on the common chord [AB] produced. Prove that the tangents [CS] and [CT] are equal in length.



- 7 [AXB] and [CXD] are two intersecting line segments. Prove that points A, B, C and D are concyclic when:

- $AX = 8$ cm, $BX = 7$ cm, $CX = 14$ cm and $DX = 4$ cm
- $AX = 5$ cm, $BX = 3.2$ cm, $CX = 8$ cm and $DX = 2$ cm.

- 8 [XAB] and [XC] are two intersecting straight line segments.

Given that $BX = 6.4$ m, $AB = 5.5$ m and $XC = 2.4$ m, prove that [OX] is a tangent to the circle through A, B and C.

- 9 Point P is 7 cm from a circle's centre. The circle has radius 5 cm. A secant is drawn from P which cuts the circle at A and B, A being closer to P. If $AB = 5$ cm, find the length of [AP].

- 10 Triangle PQR has altitudes PA, QB and RC which meet at H. Prove that:

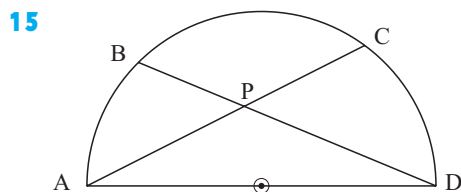
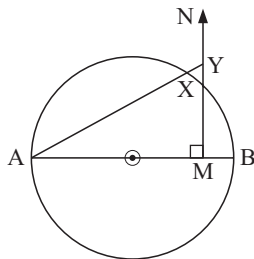
- $PH \cdot PA = PB \cdot PR$
- $PH \cdot HA = QH \cdot HB = RH \cdot HC$

- 11 Two circles have a common chord [CD]. [AB] is a common tangent to the circles. [DC] produced meets [AB] at X. Prove that X bisects [AB].

- 12 Two circles meet at P and Q. X lies on [PQ] produced. Line [XAB] is drawn to cut the first circle at A and B. Likewise line [XCD] is drawn to cut the second circle at C and D. Prove that ACDB is a cyclic quadrilateral.

- 13** Two non-intersecting circles are cut by a third circle. The first circle is cut at A and B. The second circle is cut at C and D. When the common chords are extended, they meet at X. Prove that the tangents from X to all three circles are equal in length.

- 14** [AB] is a fixed diameter of a circle and [MN] is a fixed perpendicular to [AB]. A line from point A cuts the circle at X and meets [MN] at Y. X is a moving point and consequently Y moves on [MN]. Prove that AX.AY is constant.



ABCD is a semi-circle with a diameter [AB]. P is the point of intersection of [AC] and [BD]. Prove that:

$$AP.AC + DP.DB = AD^2$$

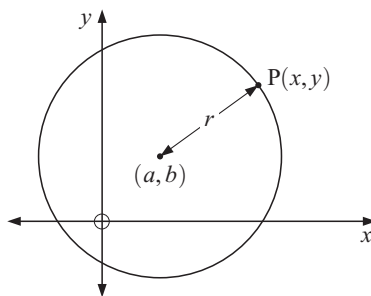
E

THE EQUATION OF A CIRCLE

Recall that if a circle has centre (a, b) and radius r and $P(x, y)$ is any point on the circle, then the **equation of the circle** is

$$(x - a)^2 + (y - b)^2 = r^2.$$

The proof is a simple application of the *distance formula*.



EXERCISE E.1

- State the coordinates of the centre and find the radius of the circle with equation:
 - $(x-2)^2 + (y-3)^2 = 4$
 - $x^2 + (y+3)^2 = 9$
 - $(x-2)^2 + y^2 = 7$
- Write down the equation of the circle with:
 - centre $(2, 3)$ and radius 5 units
 - centre $(-2, 4)$ and radius 1 unit
 - centre $(4, -1)$ and radius $\sqrt{3}$ units
 - centre $(-3, -1)$ and radius $\sqrt{11}$ units
- Find the equations of the following circles, giving your answer in the form $(x - a)^2 + (y - b)^2 = r^2$:
 - centre $(3, -2)$ and touching the x -axis
 - centre $(-4, 3)$ and touching the y -axis
 - centre $(5, 3)$ and passing through $(4, -1)$
 - ends of a diameter $(-2, 3)$ and $(6, 1)$
 - radius $\sqrt{7}$ and concentric with $(x + 3)^2 + (y - 2)^2 = 5$.

4 What do the following equations represent in 2-D coordinate geometry?

a $(x + 2)^2 + (y - 7)^2 = 5$

b $(x + 2)^2 + (y - 7)^2 = 0$

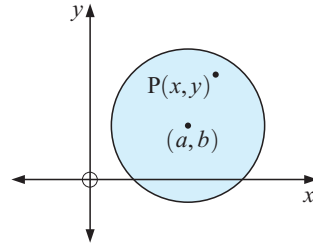
c $(x + 2)^2 + (y - 7)^2 = -5$

5 Consider the shaded region inside the circle, centre (a, b) , radius r units.

Let $P(x, y)$ be any point inside the circle.

a Show that $(x - a)^2 + (y - b)^2 < r^2$.

b What region is defined by the inequality $(x - a)^2 + (y - b)^2 > r^2$?



6 Without sketching the circle with equation $(x + 2)^2 + (y - 3)^2 = 25$, determine whether the following points lie on the circle, inside the circle or outside the circle:

a $A(2, 0)$

b $B(1, 1)$

c $C(3, 0)$

d $D(4, 1)$

Example 8

Find k if $(k, 2)$ lies on the circle with equation $(x - 2)^2 + (y - 5)^2 = 25$.

Since $(k, 2)$ lies on the circle, $x = k$ and $y = 2$ satisfy the equation.

$$\therefore (k - 2)^2 + (2 - 5)^2 = 25$$

$$\therefore (k - 2)^2 + 9 = 25$$

$$\therefore (k - 2)^2 = 16$$

$$\therefore k - 2 = \pm 4 \quad \text{and} \quad \therefore k = 6 \text{ or } -2.$$

7 Find k given that:

a $(3, k)$ lies on the circle with equation $(x + 1)^2 + (y - 2)^2 = 25$

b $(k, -2)$ lies on the circle with equation $(x + 2)^2 + (y - 3)^2 = 36$

c $(3, -1)$ lies on the circle with equation $(x + 4)^2 + (y + k)^2 = 53$.

THE GENERAL FORM OF THE EQUATION OF A CIRCLE

Notice that for

$$(x - 2)^2 + (y + 3)^2 = 7,$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 7$$

$$\therefore x^2 + y^2 - 4x + 6y + 6 = 0$$

which is of the form

$$x^2 + y^2 + Ax + By + C = 0 \quad \text{with} \quad A = -4$$

$$B = 6$$

$$C = 6.$$

In fact all circle equations can be put into this form, called the **general form**,

i.e., the general form of the equation of a circle is $x^2 + y^2 + Ax + By + C = 0$.

Often we are given equations in general form and need to find the centre and radius of the circle.

We can do this by ‘completing the square’ on both the x and y terms.

Before attempting to find the centre and radius of a circle given in the general form it is essential to make the coefficients of x^2 and y^2 be 1 (if they are not already 1).

For example, if $2x^2 + 2y^2 + 4x + 8y - 3 = 0$ then $x^2 + y^2 + 2x + 4y - \frac{3}{2} = 0$.

Consider the following example:

Example 9

Find the centre and radius of the circle with equation $x^2 + y^2 + 6x - 2y - 6 = 0$ by ‘completing the square’.

$$\begin{aligned} x^2 + y^2 + 6x - 2y - 6 &= 0 \\ \therefore x^2 + 6x + y^2 - 2y &= 6 \\ \therefore x^2 + 6x + 3^2 + y^2 - 2y + 1^2 &= 6 + 3^2 + 1^2 \quad \{\text{completing the squares}\} \\ \therefore (x + 3)^2 + (y - 1)^2 &= 16 \\ \therefore \text{the circle has centre } (-3, 1) \text{ and radius } 4 \text{ units.} \end{aligned}$$

EXERCISE E.2

- 1 Find the centre and radius of:

a $x^2 + y^2 + 6x - 2y - 3 = 0$	b $x^2 + y^2 - 6x - 2 = 0$
c $x^2 + y^2 + 4y - 1 = 0$	d $x^2 + y^2 + 4x - 8y + 3 = 0$
e $x^2 + y^2 - 4x - 6y - 3 = 0$	f $x^2 + y^2 - 8x = 0$

- 2 Find k given that:

a $x^2 + y^2 - 12x + 8y + k = 0$ is a circle with radius 4 units
b $x^2 + y^2 + 6x - 4y = k$ is a circle with radius $\sqrt{11}$ units
c $x^2 + y^2 + 4x - 2y + k = 0$ represents a circle.

- 3 Find the equation and nature of the locus of $P(x, y)$ for $A(1, 0)$ and $B(5, 0)$ given that

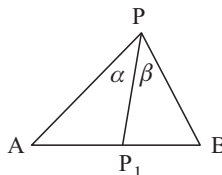
$\frac{AP}{BP} = k$, a constant where **a** $k = 3$ **b** $k = \frac{1}{3}$ **c** $k = 1$.

- 4 A is (2, 0) and B is (6, 0) and $P(x, y)$ moves such that $\frac{AP}{BP} = 2$ for all positions of P

- a** Deduce that P lies on a circle and find the circle’s centre and radius.
b If the circle from **a** cuts the x -axis at P_1 and P_2 where P_2 is to the right of P_1 , deduce the coordinates of P_1 and P_2 .

- c** If $\frac{AP}{BP} = \frac{AP_1}{BP_1}$, use the Sine Rule to deduce that $\alpha = \beta$.

- d** Prove that PP_1 bisects angle APB for all positions of P and that PP_2 bisects the exterior angle APB for all positions of P.



THE POWER OF POINT M RELATIVE TO A CIRCLE

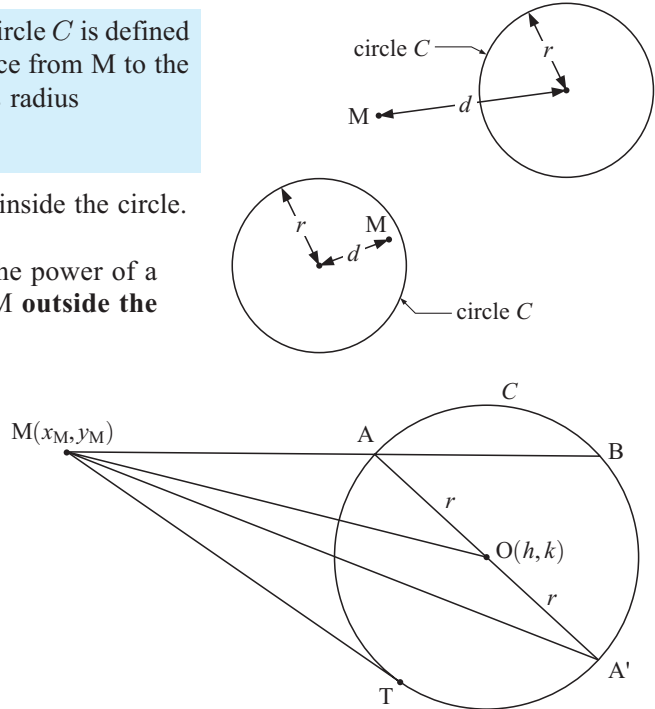
The **power** of point M relative to circle C is defined as $d^2 - r^2$ where d is the distance from M to the circle's centre and r is the circle's radius

i.e., $\text{Power } M_C = d^2 - r^2$.

Note: M could be outside, on or inside the circle.

Equivalent definitions exist for the power of a point with respect to a circle for M **outside the circle**.

Consider the diagram alongside. For the circle C, of centre O and radius r and M as shown, the following are equivalent definitions of the non-negative power of M with respect to circle C.



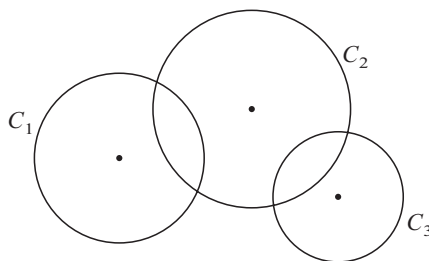
- Power $M_C = d^2 - r^2$ where $d = MO$.
- Power $M_C = MT^2$ where $[MT]$ is the tangent length.
- Power $M_C = \overrightarrow{MA} \bullet \overrightarrow{MB}$. $\{\overrightarrow{MA} \bullet \overrightarrow{MB}$ is the scalar product of $\overrightarrow{MA}$ and $\overrightarrow{MB}\}$
- Power $M_C = \overrightarrow{MA} \bullet \overrightarrow{MA'}$ where A and A' are diametrically opposite.
- Power $M_C = (x_M - h)^2 + (y_M - k)^2 - r^2$ where M is (x_M, y_M) and O is (h, k) .

EXERCISE E.3

- 1 Deduce the equivalence of all five definitions for non-negative Power M_C .
- 2 Give evidence which shows that:
 - a If Power $M_C = 0$, then M is on the circle C.
 - b If Power $M_C > 0$, then M is outside the circle.
 - c If Power $M_C < 0$, then M is inside the circle.
- 3 Explain why Power M_C depends only on the position of M relative to the circle C and the radius of C.
- 4 Two circles C_1 and C_2 intersect. M is exterior to both circles.
If Power $M_{C_1} = \text{Power } M_{C_2}$, where does M lie? Prove your conjecture.
- 5 Two circles C_1 and C_2 touch each other externally at K. M is exterior to both C_1 and C_2 . Where does M lie if Power $M_{C_1} = \text{Power } M_{C_2}$? Prove your conjecture.

- 6 For the diagram shown, is it possible to find a point M such that

$$\begin{aligned}\text{Power } M_{C_1} &= \text{Power } M_{C_2} \\ &= \text{Power } M_{C_3}?\end{aligned}$$

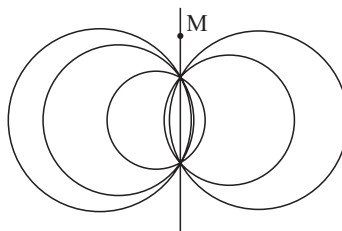


- 7 a In the given diagram, what is true about point M with respect to all of the circles?

- b Describe the locus of M as we vary the value of Power M_C .

This locus is known as the **radical axis** of the family of **coaxial circles**.

- c Does a system of circles that are mutually tangential at the same point have a radical axis? (Draw the diagram.)
- d Do two non-intersecting circles have a radical axis? Consider question 6.



Hint: Consider this argument.

For two non-intersecting circles with point M external to both, Power M_{C_1} and Power M_{C_2} are defined. So either Power $M_{C_1} > \text{Power } M_{C_2}$, or Power $M_{C_1} < \text{Power } M_{C_2}$ or Power $M_{C_1} = \text{Power } M_{C_2}$.

Consider the last case and vary M so that equality holds.

- 8 Find the power of M and its position relative to the circle C for:
- M(2, 1) and C being $(x - 3)^2 + (y + 1)^2 = 5$
 - M(3, 1) and C being $(x + 2)^2 + (y - 4)^2 = 11$
 - M(-2, 4) and C being $(x + 1)^2 + (y - 2)^2 = 8$
- 9 For $(x - 2)^2 + (y - 2)^2 = 4$ and $x^2 + y^2 = 2$:
- Find the points of intersection.
 - Find the equation of the common chord.
 - Find the equation of the radical axis of the two circles.
 - Subtract the equation of the second circle from that of the first. What do you notice?
- 10 Repeat 9 for the circles $(x - 2)^2 + y^2 = 10$ and $(x + 1)^2 + (y + 1)^2 = 20$.
- 11 a Find the equations of the radical axes of the circles $(x + 2)^2 + (y + 4)^2 = 17$ and $(x - 1)^2 + (y - 5)^2 = 5$. Do these equations intersect?
- b Find the equations of the radical axes of the circles $2x^2 + 4x + 2y^2 - 8y - 17 = 0$ and $x^2 + y^2 - 3y - 5 = 0$. Be careful!
- c Find the equations of the radical axes of the circles $(x - 1)^2 + (y - 8)^2 = 17$ and $(x - 1)^2 + (y - 5)^2 = 5$, and of the circles $(x - 1)^2 + (y - 5)^2 = 5$ and $(x + 1)^2 + (y + 1)^2 = 20$. Hence find where these radical axes intersect. (The radical centre.)

F

CONCURRENCY IN A TRIANGLE

INVESTIGATION 2

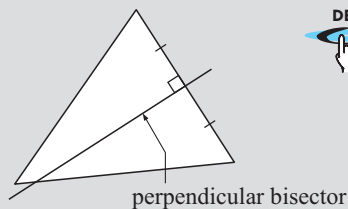
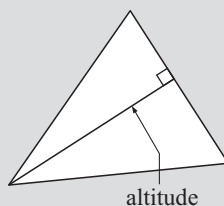
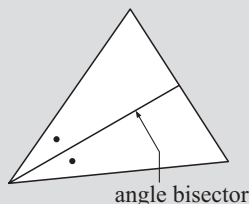
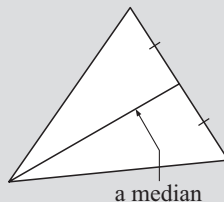
THE DIFFERENT CENTRES OF A TRIANGLE



Throughout this investigation we will refer to triangle ABC and use it to infer general properties about any given triangle. We will draw a general triangle ABC and vary it by moving the vertices and observe what happens.

What to do:

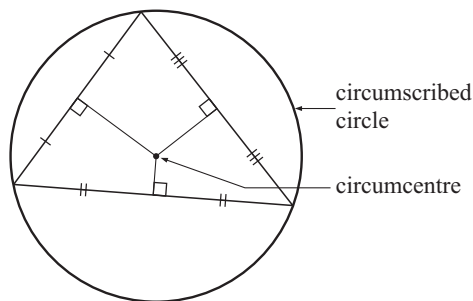
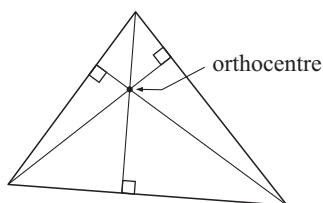
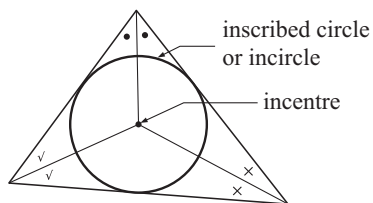
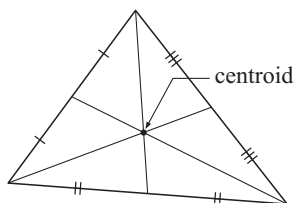
- 1 A *median* of a triangle is any line segment from a vertex to the midpoint of the opposite side. Click on the icon. Follow the instructions. Write down any observations/conclusions. Do not forget to vary the original triangle by clicking on any vertex and dragging it.
- 2 Click on the icon and follow the instructions. This interactive enables us to examine the properties of the *angle bisectors* of a triangle. Follow the instructions. Write down any observations/conclusions. Vary the triangle.
- 3 Click on the icon and follow the instructions. We draw the three *altitudes* of the triangle. Follow the instructions. Make sure you examine obtuse angled triangles as well. Write down any observations/conclusions.
- 4 Click on the icon and follow the instructions. We draw all three *perpendicular bisectors* of the sides of the triangle. Follow the instructions. Make sure you examine obtuse angled triangles as well. Write down any observations/conclusions.



From the Investigation you should have observed the following.

- The medians of a triangle are concurrent at a point which divides each median in the ratio 2 : 1.
- The angle bisectors of a triangle are concurrent. A circle can be drawn, with centre the point of concurrency, which is tangential to the sides of the triangle.
- The altitudes of a triangle are concurrent.
- The perpendicular bisectors of the sides of a triangle are concurrent. A circle can be drawn, with centre the point of concurrency, which passes through the triangles' vertices.

TERMINOLOGY

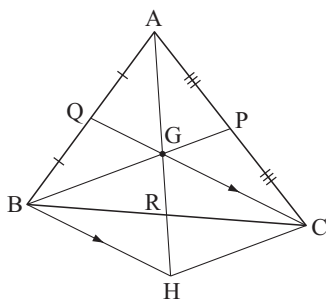


We now formally state these results as theorems, and give proofs.

Theorem: The medians of a triangle are concurrent at a point which divides each median in the ratio 2 : 1.

Note: The centroid is the point of trisection of each median.

Proof:



We draw $\triangle ABC$ and let P be the midpoint of [AC] and Q be the midpoint of [AB]. We let [BP] and [CQ] intersect at G.

We now have to prove that:

$BR = RC$, $GR = \frac{1}{3}AR$, $GQ = \frac{1}{3}CQ$ and $GP = \frac{1}{3}BP$.

We now draw [BH] parallel to [QC] to meet [AR] produced at H, and then join [CH].

Now in $\triangle ABH$, [QG] $\parallel$ [BH] and as Q is the midpoint of [AB] then G is the midpoint of [AH]. {the Midpoint theorem, converse}

Thus, [GP] is the line joining the midpoints of two sides of $\triangle AHC$
 $\Rightarrow$ [GP] $\parallel$ [HC] {the Midpoint theorem}

Hence, [BG] $\parallel$ [HC].

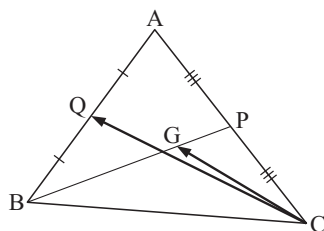
Consequently, BGCH is a parallelogram and $BR = RC$ as its diagonals bisect each other.

If $RG = a$ units, then $RH = a$ units and so $AG = GH = 2a$ units.

$\therefore AR = 3a$ units and $RG = \frac{1}{3}AR$.

Similarly, we can show $GQ = \frac{1}{3}CQ$ and $GP = \frac{1}{3}BP$.

A neat proof using vectors can be obtained using the following diagram:



We suppose G lies on median $[BP]$ such that $BG : GP = 2 : 1$.

If $\vec{CQ} = \mathbf{c}$ and $\vec{BC} = \mathbf{a}$, find in terms of $\mathbf{a}$ and $\mathbf{c}$, vector expressions for $\vec{BP}$, $\vec{BG}$, $\vec{CG}$ and $\vec{CQ}$.

Make a deduction from $\vec{CG}$ and $\vec{CQ}$.

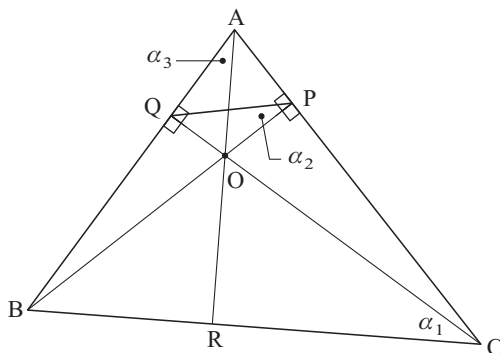
Theorem: The angle bisectors of a triangle are concurrent (at a point called the circle's incentre) and a circle with this centre can be inscribed in the triangle.

Theorem: The perpendicular bisectors of the sides of a triangle are concurrent (at a point called the circumcentre) and a circle with this centre can be drawn through the triangle's vertices.

These two theorems will be proved in the following exercise.

Theorem: The three altitudes from vertices to opposite sides of a triangle are concurrent.

Proof:



We draw two of the altitudes $[BP]$ and $[CQ]$. If they meet at O , we draw $[AO]$ and produce it to meet $[BC]$ at R .

We now need to prove that $[AR] \perp [BC]$. We join $[PQ]$.

Since $[BC]$ subtends equal angles at P and Q then $BCPQ$ is a cyclic quadrilateral {cyclic quadrilateral theorem}

$$\Rightarrow \alpha_1 = \alpha_2 \quad \{\text{angles in same segment theorem}\}$$

But $APOQ$ is a cyclic quadrilateral as its opposite angles at P and Q are supplementary (both right angles)

$$\Rightarrow \alpha_2 = \alpha_3 \quad \{\text{angles in the same segment theorem}\}$$

$$\Rightarrow \alpha_1 = \alpha_3$$

$$\Rightarrow [QR] \text{ subtends equal angles at } C \text{ and } A$$

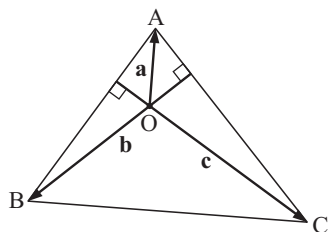
$$\Rightarrow QRCA \text{ is a cyclic quadrilateral.}$$

Thus, AC subtends equal angles at Q and R and as the angle at Q is a right angle then $\angle ARC$ is a right angle.

$$\text{i.e., } [AR] \perp [RC]$$

$$\text{Thus } [AR] \perp [BC].$$

Once again a neat proof using vectors can be obtained:



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

Use 'dot' product to show that $\mathbf{a} \bullet \mathbf{c} = \mathbf{b} \bullet \mathbf{c}$
and $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c}$.

Hence prove that $[OA] \perp [BC]$.

SIMSON'S LINE

INVESTIGATION 3



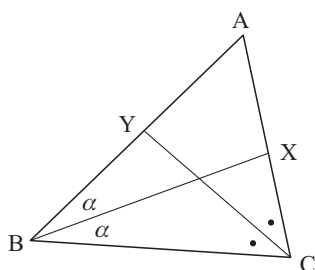
Click on the icon to investigation **Simson's Line**.

SIMSON'S LINE



EXERCISE F

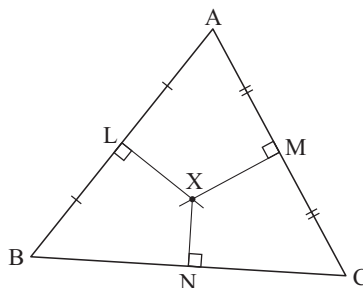
1



- How many circles can be drawn with centres on $[BX]$ and touching $[BA]$ and $[BC]$?
- How many circles can be drawn with centres on $[CY]$ and touching $[CB]$ and $[CA]$?
- What can you conclude from **a** and **b**? **d** Write out a formal proof of 'the angle bisectors of a triangle theorem'.

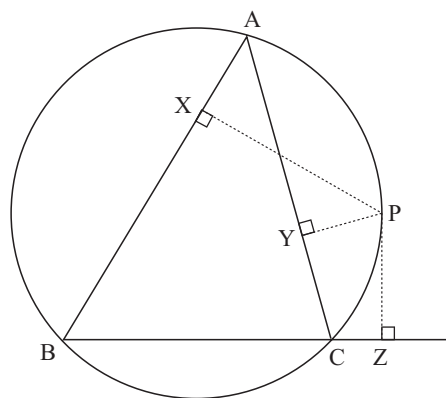
2 Consider the given figure.

- What can you conclude from $\triangle ABX$?
- What can you conclude from $\triangle ACX$?
- What do **a** and **b** tell us about:
 - point X
 - BN and NC ?
- Write out a formal proof of 'the perpendicular bisector of the sides of a triangle theorem'.



- X is the midpoint of side $[CD]$ of parallelogram ABCD and $[BX]$ meets $[AC]$ at Y. Prove that DY produced bisects $[BC]$.
- Prove that if two medians of a triangle are equal then the triangle is isosceles.
- Through the centroid of a triangle, lines are drawn parallel to two sides of the triangle. Prove that these lines trisect the third side.
- Triangle ABC has centroid G. $[AX]$ is a median of the triangle. Prove that $\triangle GBX$ has one-sixth the area of $\triangle ABC$.

- 7 A circle with centre O has diameter $[AB]$. P is a point outside the circle such that $AP = AB$. If $[PB]$ cuts the circle at R and $[OP]$ and $[AR]$ meet at X , prove that $[XP]$ has length twice that of $[OX]$.
- 8 Two circles of equal radius touch externally at B . $[AB]$ is the diameter of one circle and $[CD]$ is any diameter of the other circle. Prove that $[CB]$ produced bisects $[AD]$.
- 9 Triangle PQR is drawn. Through its vertices lines are drawn which are parallel to the opposite sides of the triangle. The new triangle formed is $\triangle ABC$. Prove that $\triangle ABC$ and PQR have the same centroid.
- 10 G is the centroid of $\triangle PQR$. A , B and C are the midpoints of $[PG]$, $[QG]$ and $[RG]$. Prove that G is the centroid of $\triangle ABC$.
- 11 Triangle ABC has centroid G . Given that $[BC]$ is fixed and A moves such that $\angle CGB$ is always a right angle, find the locus of A .
- 12 Prove that if two angle bisectors of a triangle are equal in length then the triangle is isosceles.
- 13 $PQRS$ is a rhombus. $[PM]$ is perpendicular to $[QR]$ and meets $[QS]$ at Y and $[QR]$ at M . Prove that $[RY]$ is perpendicular to $[PQ]$.
- 14 P is any point on the circumcircle of $\triangle ABC$ other than at A , B or C . Altitudes $[PX]$, $[PY]$ and $[PZ]$ are drawn to the sides of $\triangle ABC$ (or the sides produced). Prove that X , Y and Z are collinear. $[XYZ]$ is known as **Simson's line**.
- 15 $PQRS$ is a parallelogram and A , B are the orthocentres of triangles PQR and PSR . Prove that $PARB$ is also a parallelogram.
- 16 The circumcentre of a triangle is located and is point A . The midpoints of the sides of the triangle are joined to form another triangle and B , the orthocentre of this triangle is located. Prove that A and B are coincide.
- 17 Triangle ABC has altitudes $[AX]$ and $[BY]$. P and Q are the midpoints of $[AC]$ and $[BC]$ respectively. Prove that points P , Q , X and Y are concyclic.
- 18 $[AP]$ and $[BQ]$ are altitudes of $\triangle ABC$ to the opposite sides and O is the orthocentre of the triangle. X and Y are the midpoints of $[AB]$ and $[OC]$ respectively. Prove that $[XY]$ bisects $[PQ]$ at right angles.
- 19 Two circles of the same diameter meet at A and B . A third circle of the same diameter passes through B and meets the other two circles at X and Y . Prove that B is the orthocentre of $\triangle AXY$.
- 20 Triangle PQR has orthocentre O and $[RS]$ is a diameter of the circumcircle of the triangle. Prove that $SQOP$ is a parallelogram.



- 21** A triangle has a fixed base [BC] and a fixed vertical angle A. Find the locus of its orthocentre if the vertical angle is: **a** acute **b** obtuse.

G

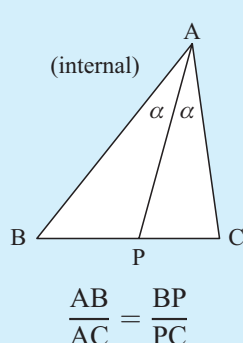
FURTHER THEOREMS

Following are several theorems which link angles and lengths in a triangle.

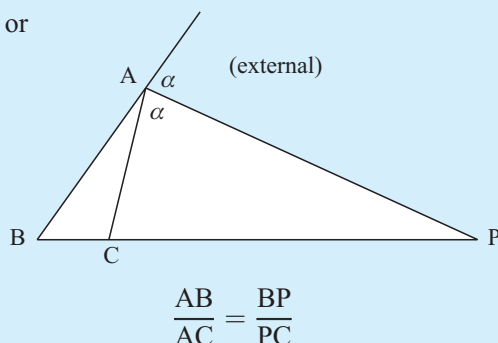
THE ANGLE BISECTOR THEOREM (OF APOLLONIUS)

Theorem: The bisectors of the angles of a triangle divide the opposite side in the same ratio of the sides containing that angle,

i.e., for

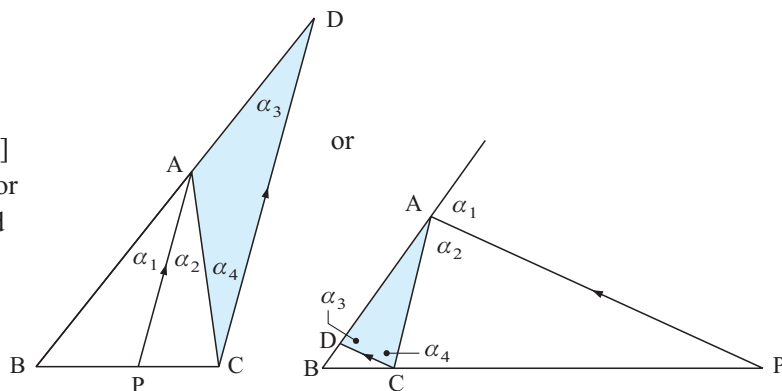


or



Proof 1: (Classical)

We draw [CD] parallel to [PA] to meet [BA] or [BA] produced at D.



$$\alpha_1 = \alpha_2 \quad \{\text{given}\}$$

$$\alpha_2 = \alpha_4 \quad \{\text{alternate angles}\}$$

$$\alpha_1 = \alpha_3 \quad \{\text{corresponding angles}\}$$

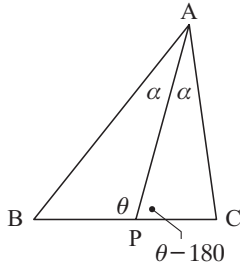
$$\therefore \alpha_3 = \alpha_4$$

$\therefore \triangle ACD$ is isosceles

$$\therefore AD = AC \quad \{\text{isosceles } \triangle \text{ theorem}\} \quad \dots\dots (1)$$

$$\text{In } \triangle BCD, \text{ as } AP \parallel DC \text{ then } \frac{BP}{PC} = \frac{BA}{AD} = \frac{AB}{AC}$$

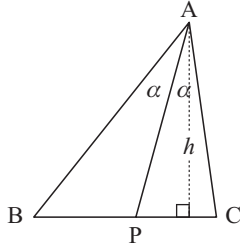
$$\text{But from (1) } AD = AC, \quad \therefore \frac{BP}{PC} = \frac{AB}{AC} \quad \{\text{QED}\}$$

Proof 2: (Internal case only)Using the Sine Rule in Δ s ABP, ACP

$$\frac{\sin \theta}{AB} = \frac{\sin \alpha}{BP} \quad \text{and} \quad \frac{\sin(180 - \theta)}{AC} = \frac{\sin \alpha}{PC}$$

But $\sin \theta = \sin(180 - \theta)$

$$\therefore \frac{AB \sin \alpha}{BP} = \frac{AC \sin \alpha}{PC} \quad \text{and so} \quad \frac{AB}{AC} = \frac{BP}{PC} \quad \{\text{QED}\}$$

Proof 3: (Internal case only)

also

$$\text{Area } \Delta ABP = \frac{1}{2} \cdot AB \cdot AP \cdot \sin \alpha = \frac{1}{2} \cdot BP \cdot h \quad \dots (1)$$

Likewise

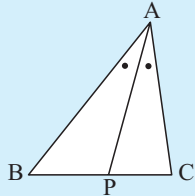
also

$$\text{Area } \Delta APC = \frac{1}{2} \cdot AP \cdot AC \cdot \sin \alpha = \frac{1}{2} \cdot PC \cdot h \quad \dots (2)$$

$$\text{Dividing (1) by (2) gives} \quad \frac{AB}{AC} = \frac{BP}{PC} \quad \{\text{QED}\}$$

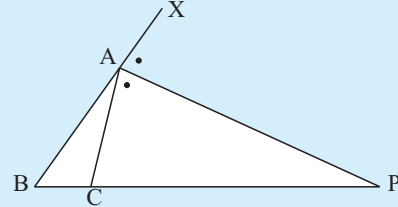
THE CONVERSE TO THE ANGLE BISECTOR THEOREM

Given



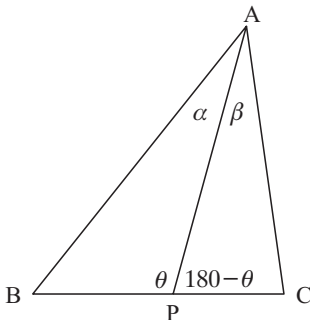
$$\text{then, if } \frac{AB}{AC} = \frac{BP}{PC},$$

or



or

$$\begin{aligned} \angle BAP &= \angle CAP \quad \{\text{internal case}\} \\ \angle XAP &= \angle CAP \quad \{\text{external case}\} \end{aligned}$$

Proof: (Internal case) Using the Sine Rule

$$\frac{\sin \alpha}{BP} = \frac{\sin \theta}{AB} \quad \text{and} \quad \frac{\sin \beta}{PC} = \frac{\sin(180 - \theta)}{AC}$$

But $\sin(180 - \theta) = \sin \theta$

$$\Rightarrow \frac{AC}{PC} \sin \beta = \frac{AB}{BP} \sin \alpha$$

$$\Rightarrow \frac{BP}{PC} \sin \beta = \frac{AB}{AC} \sin \alpha$$

$$\Rightarrow \sin \alpha = \sin \beta \quad \{\text{given } \frac{AB}{AC} = \frac{BP}{PC}\}$$

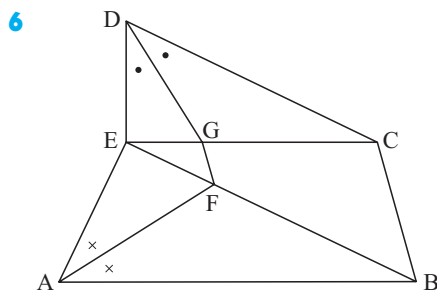
$$\Rightarrow \alpha = \beta \quad \text{or} \quad \alpha = 180 - \beta$$

$$\Rightarrow \alpha = \beta \quad \text{or} \quad \alpha + \beta = 180$$

$$\Rightarrow \alpha = \beta \quad \{\text{as } \alpha + \beta < 180\} \quad (\text{QED})$$

EXERCISE G.1

- 1 Use the Sine Rule to prove the external case of the Angle Bisector Theorem.
- 2 Use the Sine Rule to prove the external case of the *converse* to the Angle Bisector Theorem.
- 3 Prove that for a given angle of a triangle, the angle between the internal angle bisector and the external angle bisector is a right angle.
- 4 Draw any straight line [AB]. Divide [AB] in the ratio 5 : 3 using a compass-ruler construction which includes the use of the angle bisector theorem.
- 5 Triangle ABC is isosceles as $AB = AC$. Suppose P is any point within the triangle. The bisector of angle PAB meets [BP] at H. The bisector of angle CAP meets [CP] at K. Prove that [HK] is parallel to [BC].



EAB and EDC are similar triangles with corresponding vertices in that order.

[AF] bisects angle BAE and meets [BE] at F.

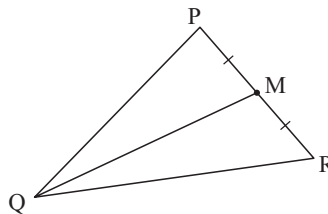
[DG] bisects angle CDE and meets [EC] at G.

Prove that [GF] is parallel to [CB].

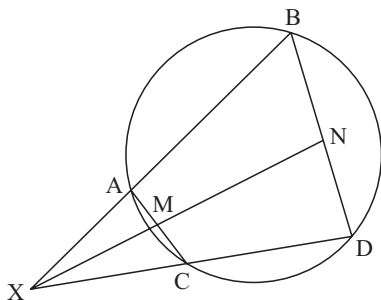
- 7 P is the midpoint of [BC] of triangle ABC. [PQ] is the bisector of angle APB and cuts AB at Q. [QR] is drawn parallel to [AB]. Prove that angle QPR is a right angle.
- 8 A semi-circle has diameter [AB]. P lies on the semi-circle and [PQ] bisects angle APB, cutting [AB] at Q. [PC] is drawn perpendicular to [AB], cutting [AB] at C.

Prove that $\frac{AQ}{QB} = \frac{AC}{PC}$.

- 9 M is the midpoint of [PR] of triangle PQR and [QM] bisects angle PQR.
 - a Prove that $\triangle PQR$ is isosceles.
 - b Why cannot congruence be used in this figure?

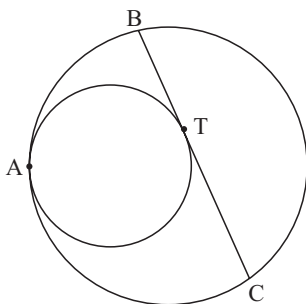


10



If XMN is the angle bisector of angle BXD, prove that $BN : ND = CM : MA$.

- 11** Triangle ABC has interior angle bisectors which meet [BC], [CA] and [AB] in points P, Q and R respectively. Prove that $AR \cdot BP \cdot CQ = AQ \cdot BR \cdot CP$.
- 12** [AP] is the exterior bisector of angle A of triangle ABC. [AP] cuts [BC] produced at P. [CB] is produced to M such that $BM = CP$. [MN] is parallel to [AP] meeting [AB] produced at N. Prove that $BN = AC$.
- 13** PQR is a triangle with vertices on a circle. [RA] is drawn so that it is parallel to the tangent at P and cuts [PQ] (or [PQ] produced) at A. The angle bisectors of angles PQR and PRA meet PR and PQ at L and M. Prove that [LM] is parallel to [RA].
- 14** Triangle PQR has median [PS]. [SA] and [SB] bisect angles PSR and PSQ respectively. (They meet [PR] and [PQ] in A and B.) Prove that [AB] is parallel to [QR].

15

Two circles touch internally at A. [BC] is a chord of the larger circle which is a tangent to the smaller one at T.

Prove that $AB : AC = BT : TC$.

- 16** Triangle PQR has incentre O. [OP] meets [RQ] at S. Prove that $PO : OS = (PQ + PR) : QR$.
- 17** A circle has diameter [PQ] and [RS] is any chord perpendicular to [PQ]. T lies on [RS]. [PT] produced and [QT] produced meet the circle at A and B respectively. Prove that any two adjacent sides of RASB are in the same ratio as the other two sides.

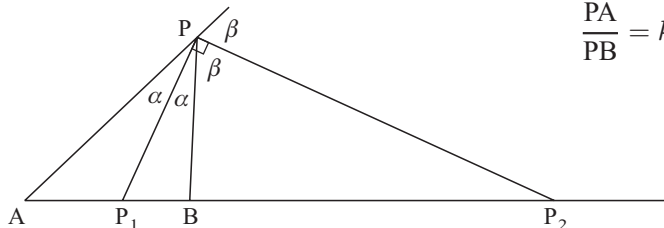
APOLLONIUS' CIRCLE THEOREM

In questions 3 and 4 of Exercise E.2 we observed that:

If A and B are fixed points such that $\frac{PA}{PB} = k$, where k is a constant, $k \neq 1$, then the locus of P is a circle.

This is **Apollonius' Circle Theorem**.

Proof:



$$\frac{PA}{PB} = k, \quad k \neq 1$$

We draw the internal and external bisectors of angle APB.

Notice that $2\alpha + 2\beta = 180^\circ$ {angles on a line}

$$\therefore \alpha + \beta = 90^\circ \text{ for all positions of P}$$

i.e., $\angle P_1 P P_2$ is a right angle.

$$\text{Now } \frac{AP_1}{BP_1} = \frac{AP}{BP} \quad \{\text{angle bisector theorem}\}$$

$$\therefore \frac{AP_1}{BP_1} = k \Rightarrow P_1 \text{ is a fixed point.}$$

$$\text{Likewise, } \frac{AP_2}{BP_2} = \frac{AP}{BP} = k \Rightarrow P_2 \text{ is a fixed point.}$$

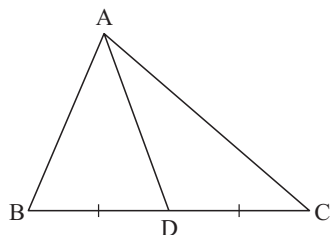
As P_1 and P_2 are fixed points and $\angle P_1 P P_2$ is a right angle, $[P_1 P_2]$ subtends a right angle at P as P moves.

Consequently, P traces out a circle, with centre the midpoint of $[P_1 P_2]$. (QED)

APOLLONIUS' MEDIAN THEOREM

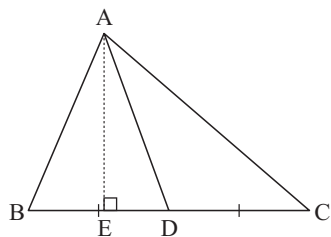
Theorem: In any triangle, the sum of the squares of two sides is equal to twice the square of half the third side plus twice the square of the median which bisects the third side.

For example,



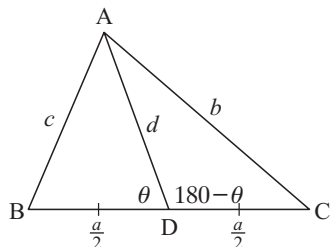
If D is the midpoint of base $[BC]$ of $\triangle ABC$,
then $AB^2 + AC^2 = 2(AD^2 + BD^2)$
or $2(AD^2 + DC^2)$.

Proof 1:



$$\begin{aligned} AB^2 + AC^2 &= BE^2 + AE^2 + CE^2 + AE^2 \quad \{\text{Pythagoras}\} \\ &= (BD - ED)^2 + (CD + ED)^2 + 2AE^2 \\ &= (BD - ED)^2 + (BD + ED)^2 + 2AE^2 \quad \{\text{as } CD = BD\} \\ &= BD^2 - 2BD \cdot ED + ED^2 + BD^2 + 2BD \cdot ED + ED^2 + 2AE^2 \\ &= 2BD^2 + 2ED^2 + 2AE^2 \\ &= 2BD^2 + 2(ED^2 + AE^2) \\ &= 2BD^2 + 2AD^2 \quad \{\text{Pythagoras}\} \quad (\text{QED}) \end{aligned}$$

Proof 2:



By the Cosine Rule

$$c^2 = \left(\frac{a}{2}\right)^2 + d^2 - 2\left(\frac{a}{2}\right)d \cos \theta \dots\dots (1)$$

$$b^2 = \left(\frac{a}{2}\right)^2 + d^2 - 2\left(\frac{a}{2}\right)d \cos(180 - \theta)$$

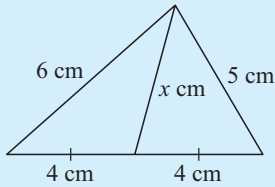
$$= \left(\frac{a}{2}\right)^2 + d^2 + 2\left(\frac{a}{2}\right)d \cos \theta \dots\dots (2)$$

$$\{\text{as } \cos(180 - \theta) = -\cos \theta\}$$

$$\text{Thus } b^2 + c^2 = 2\left(\frac{a}{2}\right)^2 + 2d^2 \quad \{\text{adding (1) and (2)}\}$$

Example 10

Find the length of the shortest median of a triangle with sides 8 cm, 6 cm and 5 cm.



$$6^2 + 5^2 = 2x^2 + 2(4)^2 \quad \{\text{Apollonius' median theorem}\}$$

$$\therefore 2x^2 = 36 + 25 - 32$$

$$\therefore 2x^2 = 29$$

$$\therefore x^2 = \frac{29}{2}$$

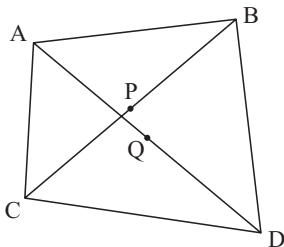
$$\therefore x = \sqrt{\frac{29}{2}} \approx 3.81$$

As the shortest median is the median to the longest side then the length required is 3.81 cm (approx.)

EXERCISE G.2

- 1 Find the length of the longest median of a triangle with sides 7 cm, 9 cm and 10 cm.
- 2 Two sides of a triangle have lengths 12 cm and 9 cm. The median to the third side has length 7 cm. Find the length of the third side.
- 3 Find the lengths of the three sides of a triangle given that its medians have lengths 3 cm, 4 cm and 5 cm.
- 4 Two sides of a parallelogram have lengths of 8 cm and 12 cm. If one diagonal is 16 cm long, find the length of the other diagonal.
- 5 Prove that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals.
- 6 If X is any point inside rectangle ABCD, prove that $AX^2 + CX^2 = BX^2 + DX^2$. Does the above result hold if X is any point outside the rectangle?
- 7 If [AB] is a fixed interval and point P moves such that $AP^2 + BP^2$ is constant, prove that the locus of P is a circle.
- 8 A circle has fixed diameter [AB] and C lies on the circle. D and E lie on [AB] such that $AD = BE$ and D, E are fixed. Prove that $CD^2 + CE^2$ is constant for all positions of C.
- 9 F is a fixed point on a circle with diameter [AB]. [PQ] is parallel to [AB] and is variable. Prove that $FP^2 + FQ^2$ is constant.
- 10 Use Apollonius' median theorem to prove that if two medians of a triangle are equal, then the triangle is isosceles.

11



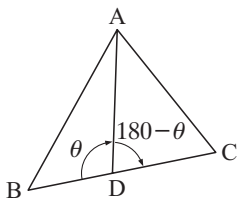
P and Q are the midpoints of the diagonals of a quadrilateral. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

STEWART'S THEOREM

If D is any point on base [BC] of triangle ABC which divides BC in the ratio $m : n$, then $nAB^2 + mAC^2 = (m+n)AD^2 + mCD^2 + nBD^2$.

Proof:



Given $BD : DC = m : n$,

by the Cosine rule,

$$AB^2 = AD^2 + BD^2 - 2 \cdot AD \cdot BD \cdot \cos \theta \dots\dots (1)$$

$$AC^2 = AD^2 + DC^2 - 2 \cdot AD \cdot DC \cdot \cos(180 - \theta)$$

$$\text{But } \cos(180 - \theta) = -\cos \theta$$

$$\therefore AC^2 = AD^2 + DC^2 + 2 \cdot AD \cdot DC \cdot \cos \theta \dots\dots (2)$$

Using (1) and (2)

$$\begin{aligned} nAB^2 + mAC^2 &= nAD^2 + nBD^2 - 2n \cdot AD \cdot BD \cdot \cos \theta \\ &\quad + mAD^2 + mDC^2 + 2m \cdot AD \cdot DC \cdot \cos \theta \\ &= (m+n)AD^2 + mDC^2 + nBD^2 + 2 \cdot AD \cdot \cos \theta \cdot (mDC - nBD) \end{aligned}$$

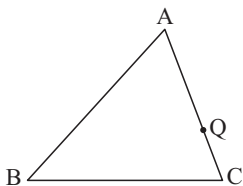
$$\begin{aligned} \text{where } \frac{BD}{DC} &= \frac{m}{n} \Rightarrow mDC = nBD \\ &\Rightarrow mDC - nBD = 0 \end{aligned}$$

$$= (m+n)AD^2 + mDC^2 + nBD^2$$

EXERCISE G.3

1 Deduce Apollonius' theorem from Stewart's theorem.

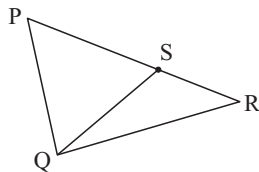
2



Q lies on [AC] such that $AQ : QC = 2 : 1$.

If $AC = 6$ cm, $BC = 7$ cm and $AB = 8$ cm, find the length of [BQ].

3



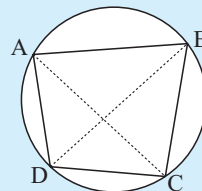
$PS : SR = 5 : 3$, $QS = QP = 9$ cm and $PR = 8$ cm. Find the length of QR.

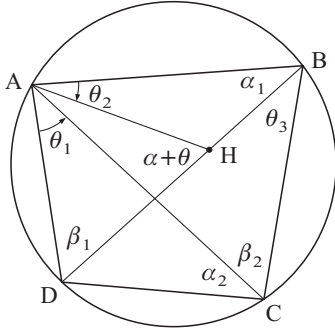
PTOLEMY'S THEOREM

Theorem:

If a quadrilateral is cyclic, then the sum of the products of the lengths of the two pairs of opposite sides is equal to the product of the diagonals,

$$\text{i.e., } AB \cdot CD + BC \cdot DA = AC \cdot BD$$



Proof:

First of all we do a construction which enables us to create similar triangles. We draw $[AH]$ where H lies on $[DB]$ such that $\theta_1 = \theta_2$ as shown.

Now in Δ s ABH, ACD

$$\theta_1 = \theta_2 \quad \{\text{construction}\}$$

$$\alpha_1 = \alpha_2 \quad \{\text{angles in same segment}\}$$

$\therefore$ the triangles are equiangular and so are similar

$$\therefore \frac{AB}{AC} = \frac{BH}{CD}$$

$$\text{So, } BH = \frac{AB \cdot CD}{AC} \dots\dots (1)$$

Notice that $\angle AHD = \alpha + \theta$ {exterior angle of ΔABH } and $\theta_1 = \theta_3$ {angles in same segment}

$\therefore \Delta$ s ADH and ACB are equiangular {each contains angles of β and $\theta + \alpha$ }

$$\frac{HD}{BC} = \frac{AD}{AC}$$

$$\therefore HD = \frac{BC \cdot DA}{AC} \dots\dots (2)$$

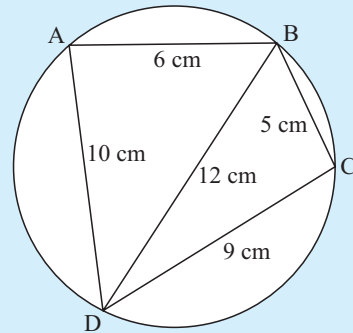
$$\text{Thus from (1) and (2), } BD = BH + HD = \frac{AB \cdot CD}{AC} + \frac{BC \cdot DA}{AC}$$

$$\therefore BD = \frac{AB \cdot CD + BC \cdot DA}{AC}$$

$$\text{Hence, } AB \cdot CD + BC \cdot DA = AC \cdot BD \quad (\text{QED})$$

Example 11

Find AC given that $[BD]$ has length 12 cm.



$$AB \cdot CD + BC \cdot DA = AC \cdot BD \quad \{\text{Ptolemy's theorem}\}$$

$$\therefore 6 \times 9 + 5 \times 10 = AC \cdot 12$$

$$\therefore 104 = AC \cdot 12$$

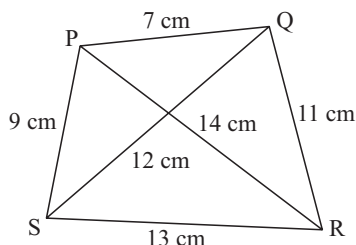
$$\therefore AC = 8\frac{2}{3}$$

i.e., AC has length $8\frac{2}{3}$ cm.

EXERCISE G.4

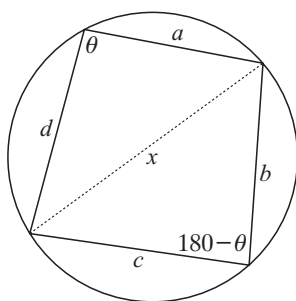
- 1 The sides of a cyclic quadrilateral given in clockwise order are 6 cm, 7 cm, 9 cm and 10 cm. If one diagonal is 8 cm, find the length of the other diagonal.
- 2 Three consecutive sides of a cyclic quadrilateral have lengths 6 cm, 7 cm and 11 cm. Its diagonals have lengths 8 cm and 12 cm. Find the length of the fourth side of the cyclic quadrilateral.

3



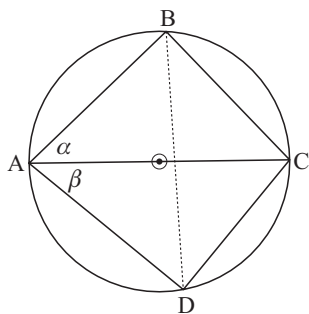
- a Is PQRS a cyclic quadrilateral?
- b What assumption(s) have you made in your argument in a?

4



- a Use the given figure and the Cosine Rule to deduce that $x^2 = \frac{(ac + bd)(ab + cd)}{(bc + ad)}$
- b If the other diagonal has length y units, deduce that $y^2 = \frac{(ac + bd)(ab + bc)}{(ab + cd)}$
- c Hence, prove Ptolemy's theorem.

5

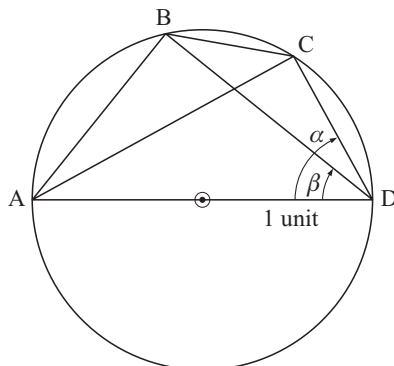


[AC] is a diameter of a circle, centre O, radius 1 unit.

If $\angle BAC = \alpha$ and $\angle DAC = \beta$, use Ptolemy's theorem to prove the addition formula.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

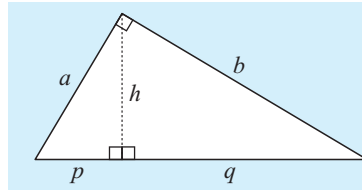
- 6 Similarly to 2, use Ptolemy's theorem and the figure alongside to prove that $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.



H

PROPORTIONALITY RIGHT ANGLED TRIANGLES

EUCLID'S THEOREM FOR PROPORTIONAL SEGMENTS IN A RIGHT ANGLED TRIANGLE



In the given figure:

- $h^2 = pq$
- $a^2 = p(p + q)$
- $b^2 = q(p + q)$

Proof:

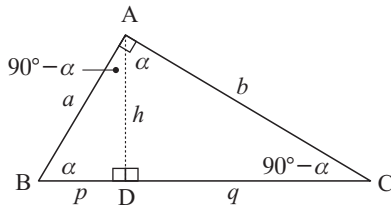
Δ s ABC, DBA and DAC are equiangular and hence are similar.

Thus, the sides are in proportion. Consequently,

$$\frac{DB}{DA} = \frac{DA}{DC} \Rightarrow \frac{p}{h} = \frac{h}{q} \Rightarrow h^2 = pq$$

$$\text{Also, } \frac{AB}{BC} = \frac{DB}{BA} \Rightarrow \frac{a}{p+q} = \frac{p}{a}$$

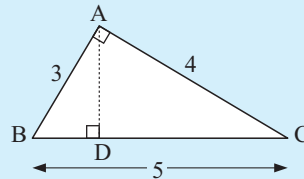
$$\therefore a^2 = p(p + q) \quad \{\text{from the first equation}\}$$



The third result follows in a similar fashion.

Example 12

Find BD and AD in:



From Euclid's theorem, $BA^2 = BD \cdot BC$

$$\therefore 3^2 = BD \times 5$$

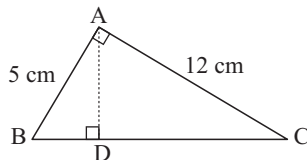
$$\therefore BD = 1.8$$

$$\text{Also } AD^2 = BD \times DC \quad \therefore AD = \sqrt{1.8 \times 3.2} = 2.4$$

Note: Equating areas of ΔABC gives $\frac{1}{2} \times 5 \times AD = \frac{1}{2} \times 3 \times 4 \Rightarrow AD = \frac{12}{5} = 2.4$

EXERCISE H.1

1

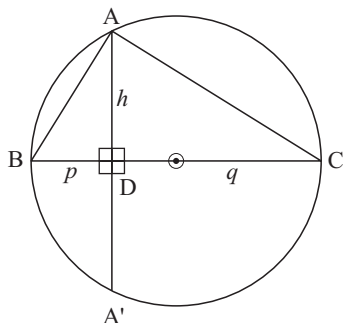


Find the length of:

- a** [BC] **b** [DC] **c** [AD].

- 2 Notice that Euclid's theorem (in a right angled triangle) was proved using similar triangles only. Use Euclid's theorem to prove that $x^2 + y^2 = (p + q)^2$, i.e., to prove Pythagoras' theorem.

3



- a** [BC] is the diameter of a circle, centre O. A lies on the circle and A' is the image of A under a reflection in the line [BC].

Use the 'products of chords' theorem to deduce that $h^2 = pq$.

- b** By drawing another circle on the diagram in **a**, prove that $a^2 = p(p + q)$ using 'the products of chords theorem - special case'.

AREA COMPARISON

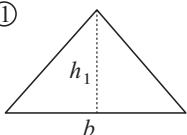
Theorem:

Areas of triangles are proportional to:

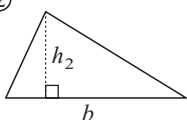
- altitudes if bases are equal
- bases if altitudes are equal
- squares of corresponding sides if the triangles are similar.

Proof:

①



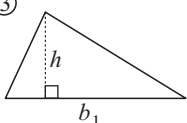
②



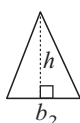
$$\frac{\text{Area of (1)}}{\text{Area of (2)}} = \frac{\frac{1}{2}bh_1}{\frac{1}{2}bh_2} = \frac{h_1}{h_2}$$

proves the first part.

③



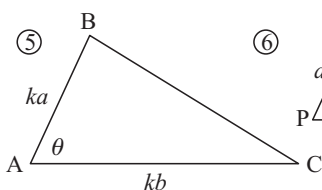
④



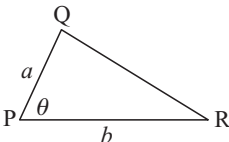
$$\frac{\text{Area of (3)}}{\text{Area of (4)}} = \frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}$$

proves the second part.

⑤



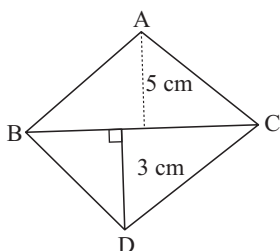
⑥



$$\begin{aligned} \frac{\text{Area of (5)}}{\text{Area of (6)}} &= \frac{\frac{1}{2}(ka)(kb) \sin \theta}{\frac{1}{2}ab \sin \theta} \\ &= k^2 \\ &= \frac{(ka)^2}{a^2} \\ &= \frac{AB^2}{PQ^2} \end{aligned}$$

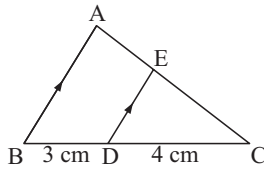
EXERCISE H.2

1



What is the ratio of the area of triangle ABC to that of BCD?

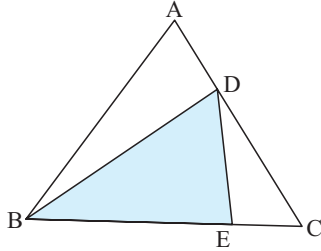
2



What is the ratio of:

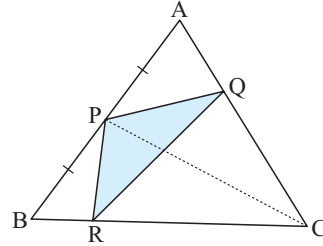
- a area $\triangle DEC$: area $\triangle ABC$
- b area $\triangle DEC$: area $ABDE$?
- c If area $ABDE = 6 \text{ cm}^2$, find the area of $\triangle ABC$.

3



D divides $[AC]$ in the ratio $1 : 2$.
E divides $[BC]$ in the ratio $3 : 1$.
What fraction of $\triangle ABC$ is $\triangle BDE$
(by area)?

4



P is the midpoint of $[AB]$.
Q divides $[AC]$ in the ratio $1 : 2$.
R divides $[BC]$ in the ratio $1 : 3$.
What fraction of $\triangle ABC$ is $\triangle PQR$
(by area)?

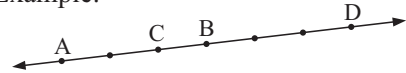
I

HARMONIC RATIOS

If A, B, C and D are collinear where C divides $[AB]$ internally in some ratio and D divides $[AB]$ externally in the same ratio then C and D divide $[AB]$ **harmonically**.

If $AC : CB = m : n$ then $m : n$ is the **harmonic ratio**.

Example:

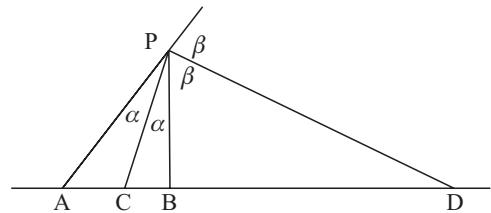


and $AD : DB = 6 : 3$ or $2 : 1$.

So, C and D divide $[AB]$ harmonically
and the harmonic ratio is $2 : 1$.

Note:

For the Apollonius Circle,
 $AC : CB = AP : BP$ and
 $AD : DB = AP : BP$.
So, $AC : CB = AD : DB$
 $\Rightarrow$ C and D divide $[AB]$ harmonically.



EXERCISE I

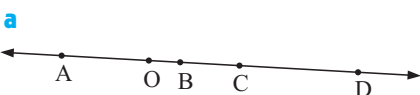
1



M divides $[PQ]$ internally in the ratio $1 : 4$.
Locate N such that M and N divide:

- a $[PQ]$ harmonically
- b $[QP]$ harmonically. Illustrate.

2

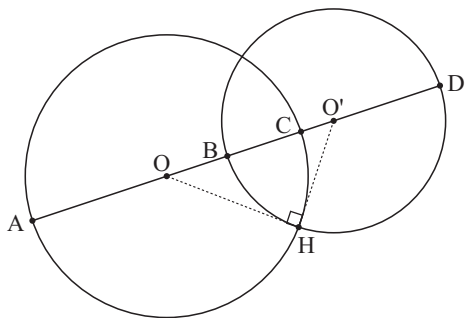


O is the midpoint of $[AC]$. B and D are points on the line such that $OC^2 = OB \cdot OD$.
Show that B and D divide $[AC]$ harmonically.

Hint: Let $OB = b$, $BC = c$ and $CD = d$.

Show that $b + c = \frac{bd}{c}$ and that $AB : BC = AD : DC$.

b



Two intersecting circles are **orthogonal** if their radii at their points of intersection are perpendicular.

Suppose we have two orthogonal circles and any diameter [AC] of one circle (say) cuts the other circle at B and D.

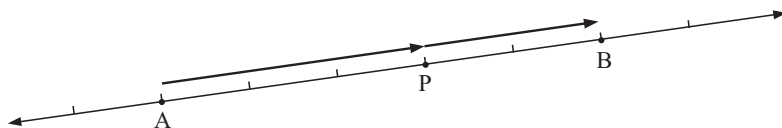
Prove that B and D divide [AC] harmonically. (**Hint:** Consider $OB \cdot OD$)

J

PROPORTIONAL DIVISION (INCLUDING DIRECTION)

INTERNAL DIVISION

Consider point P on the line through A and B where P divides [AB] *internally* in the ratio 3 : 2.



This means that
 $\overrightarrow{AP} : \overrightarrow{PB} = 3 : 2$.

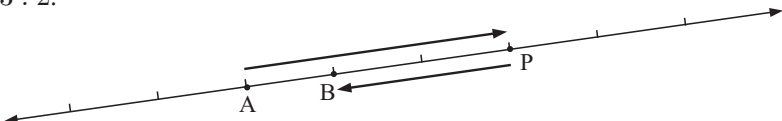
Notice that the movement from A to P and then from P to B is in the same direction.

In general, If P divides [AB] **internally** in the ratio $l : m$ then $\overrightarrow{AP} : \overrightarrow{PB} = l : m$.

$$\frac{AP}{PB} = \frac{l}{m} \quad \text{where } l \text{ and } m \text{ are positive.}$$

EXTERNAL DIVISION

Consider point P on the line through A and B where P divides [AB] *externally* in the ratio 3 : 2.



This means that
 $\overrightarrow{AP} : \overrightarrow{PB} = -3 : 2$

So,

If P divides [AB] **externally** in the ratio $l : m$ then $\overrightarrow{AP} : \overrightarrow{PB} = -l : m$

$$\frac{AP}{PB} = -\frac{l}{m} \quad \text{where } l \text{ and } m \text{ are positive.}$$

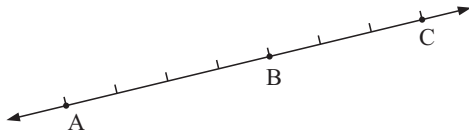
EULER'S THEOREM (4 POINTS ON A LINE)

If A, B, C and D are four points (in order) on a line then
 $AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$.

The proof of this theorem is in the following Exercise.

EXERCISE J

1



Find the ratio in which:

- a** B divides [AC] **b** B divides [CA]
c C divides [AB] **d** C divides [BA]
e A divides [BC] **f** A divides [CB]

2 Draw a line diagram to illustrate:

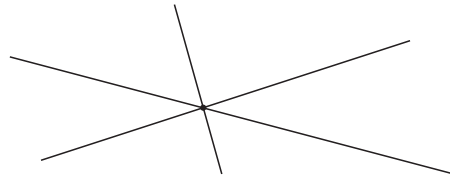
- a** A divides [BC] internally in the ratio 3 : 1
b A divides [BC] externally in the ratio 3 : 1
c L divides [PQ] internally in the ratio 4 : 3
d M divides [PQ] externally in the ratio 4 : 3.
- 3 If A divides [CB] externally in the ratio 1 : 3, in what ratio does C divide [BA]?
- 4 Prove Euler's theorem for four points on a line.

K

CONCURRENCY AND CEVA'S THEOREM

Three or more lines are **concurrent** if they intersect at a common point.

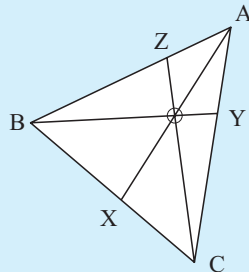
These lines are concurrent.



CEVA'S THEOREM

Any three concurrent lines drawn from the vertices of a triangle divide the sides (produced if necessary) so that the product of their respective ratios is unity.

So, for:



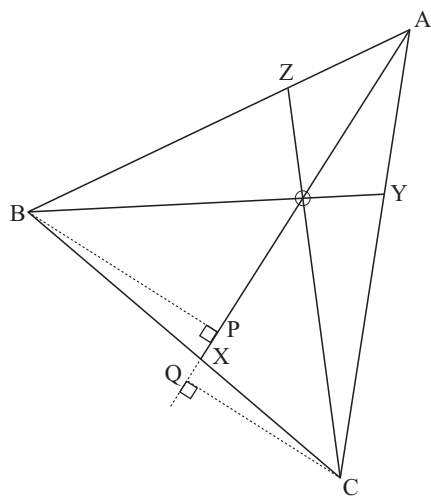
then

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$

Note: $\overset{A}{\underbrace{\quad}} \overset{B}{\underbrace{\quad}} \overset{C}{\underbrace{\quad}} \overset{A}{\underbrace{\quad}}$

helps establish the correct ratios.

Proof: The proof uses the theorem that if two triangles have the same altitude, then the ratio of their areas is the same as the ratio of their bases.



We draw altitudes [BP] for $\triangle AOB$ and [CQ] for $\triangle AOC$.

$\triangle$ s BXP and CXQ are similar.

$$\therefore \frac{BX}{CX} = \frac{BP}{CQ}$$

$$\therefore \frac{BX}{XC} = \frac{\triangle AOB}{\triangle AOC} \dots\dots (1)$$

{as have common base AO}

Similarly

$$\frac{CY}{YA} = \frac{\triangle BOC}{\triangle AOB} \dots\dots (2) \quad \text{and}$$

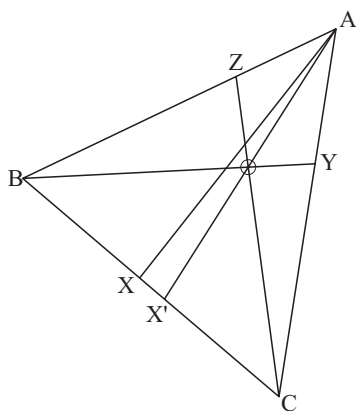
$$\frac{AZ}{ZB} = \frac{\triangle AOC}{\triangle BOC} \dots\dots (3)$$

Multiplying (1), (2) and (3) gives $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1 \quad (\text{QED})$

THE CONVERSE OF CEVA'S THEOREM

If three lines are drawn from the vertices of a triangle to cut the opposite sides (or sides produced) such that the product of their respective ratios is unity, then the three lines are concurrent.

Proof:



Let [BY] and [CZ] meet at O and [AO] produced at X'.

$$\therefore \frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

But $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1 \quad \{\text{given}\}$

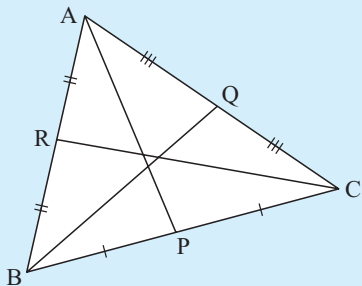
$$\therefore \frac{BX'}{X'C} = \frac{BX}{XC}$$

$\Rightarrow$ X and X' coincide

$\Rightarrow$ [AX], [BY] and [CZ] are concurrent.

Example 13

Use the converse of Ceva's theorem to prove that the medians of a triangle are concurrent.



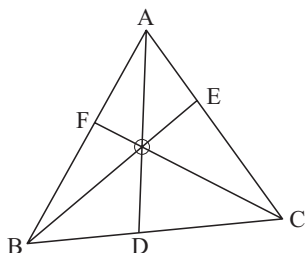
Let the medians of $\triangle ABC$ be $[AP]$, $[BQ]$ and $[CR]$ respectively.

Now $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1 \times 1 \times 1 = 1$

$$\Rightarrow [AP], [BQ] \text{ and } [CR] \text{ are concurrent.}$$

EXERCISE K

1



In $\triangle ABC$, D lies on $[BC]$ such that $BD = \frac{1}{2}BC$.

E lies on $[AC]$ such that $CE = \frac{2}{3}CA$.

F lies on $[AB]$.

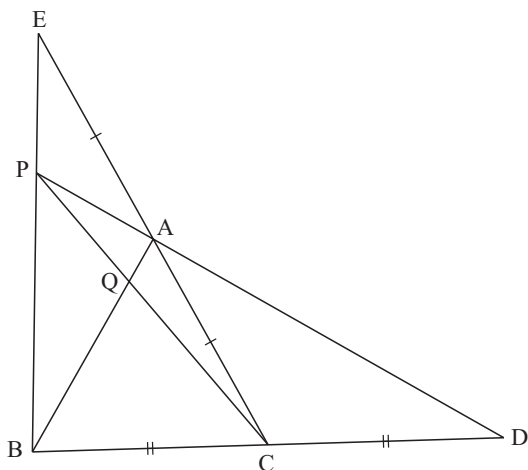
Find:

a AF : FB **b** BO : BE

area $\triangle AOB$: area $\triangle BOC$

- 2** P, Q and R lie on sides [AB], [BC] and [CA] of triangle ABC.
If $AP = \frac{2}{3}AB$, $BQ = \frac{3}{4}BC$ and $CR = \frac{1}{7}CA$, prove that [AB], [BC] and [CA] are concurrent.
- 3** Use the converse of Ceva's theorem to prove that the angle bisectors of a triangle are concurrent.

4



In the figure given $BC = CD$ and $CA = AE$.

Find the ratio in which Q divides [AB].

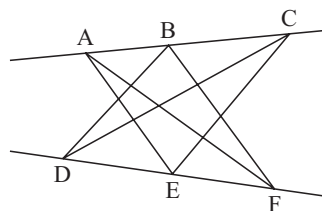
- 5 Use the converse of Ceva's theorem to prove that the altitudes from vertices of a triangle are concurrent.
- 6 Tangents to the inscribed circle of triangle PQR are parallel to [QR], [RP] and [PQ] respectively and they touch the circle at A, B and C.
Prove that [PA], [QB] and [RC] are concurrent.

L

MENELAUS' THEOREM

Many amazing discoveries have been made by people who were simply drawing figures with straight edges and compasses.

For example, **Pappus** drew two line segments and placed three points A, B and C on one of them and three points D, E and F on the other. He then joined A to E and F, B to D and F and C to D and E and made an interesting observation about some of the points of intersection. Try this for yourself using pencil and paper. Any suspicion that you might have could be written down as a **conjecture**.

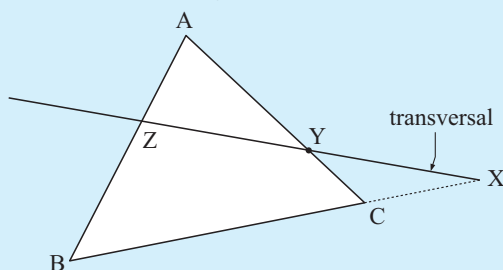


Recall also the three circles problem at the start of this chapter. The converse of **Menelaus' theorem** provides us with a method of establishing collinearity of points in certain situations.

MENELAUS' THEOREM

If a transversal is drawn to cut the sides of a triangle (produced if necessary), the product of the ratios of alternate segments is minus one.

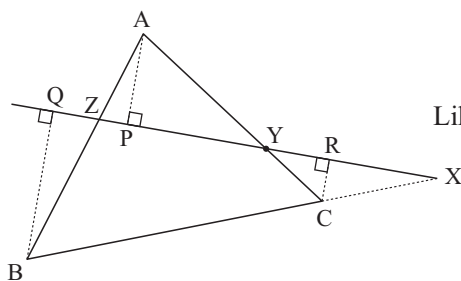
So, for



$$\text{i.e., } \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

Note: The transversal does not have to intersect the triangle.

Proof: We draw [AP], [BQ] and [CR] perpendicular to the transversal.



Δ s BQX, CRX are similar.

$$\Rightarrow \frac{BX}{XC} = \frac{BQ}{CR}$$

Likewise, $\frac{CY}{AY} = \frac{CR}{AP}$ { Δ CYR is similar to Δ AYP}

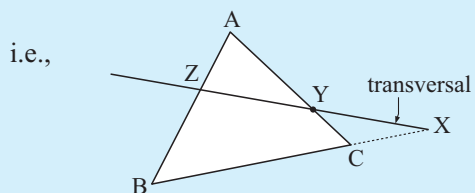
and $\frac{AZ}{BZ} = \frac{AP}{BQ}$ { Δ BQZ is similar to Δ APZ}

$$\text{Consequently, } \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{BZ} = \frac{BQ}{CR} \cdot \left(-\frac{CR}{AP}\right) \cdot \frac{AP}{BQ} = -1$$

The case where the transversal does not cut the triangle is left to the reader.

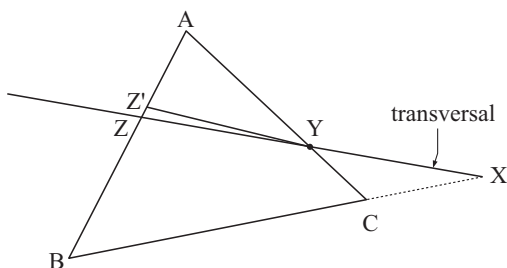
THE CONVERSE OF MENELAUS' THEOREM

If three points on two sides of a triangle and the other side produced (or on all three sides produced) are such that the product of the ratios of alternate segments is equal to minus one, then the three points are collinear.



If $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$
then X, Y and Z are collinear.

Proof: (for the case illustrated)



Let XYZ' be a straight line

$$\therefore \frac{AZ'}{Z'B} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

{Menelaus' theorem}

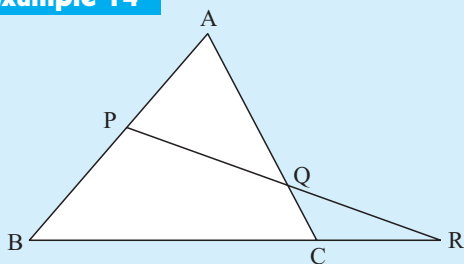
But $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$

$$\therefore \frac{AZ'}{Z'B} = \frac{AZ}{ZB}$$

So Z' and Z coincide.

$\therefore$ X, Y, Z are collinear.

Example 14



If P divides [AB] in the ratio 2 : 3 and Q divides [AC] in the ratio 5 : 2, in what ratio does R divide [BC]?

PQR is a transversal of $\triangle ABC$

$$\Rightarrow \frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{CQ}{QA} = -1$$

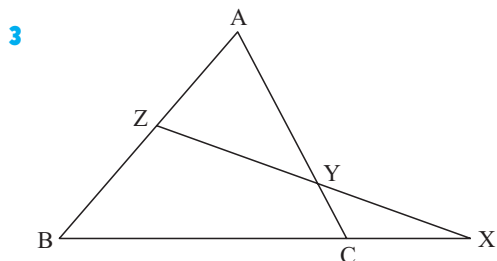
$$\Rightarrow \frac{2}{3} \times \frac{BR}{RC} \times \frac{2}{5} = -1$$

$$\Rightarrow \frac{BR}{RC} = -\frac{15}{4}$$

$\Rightarrow$ R divides [BC] externally in the ratio 15 : 4.

EXERCISE L

- 1 Transversal XYZ of triangle ABC cuts [BC], [CA] and [AB] produced in X, Y and Z. If $BX : XC = 3 : 5$ and $YA : CY = 2 : 3$, find the ratio in which Z divides [AB].
- 2 ABC is a triangle in which D divides [BC] in the ratio 2 : 3. If E divides [CA] in the ratio 5 : 4, find the ratio in which [BE] divides [AD].



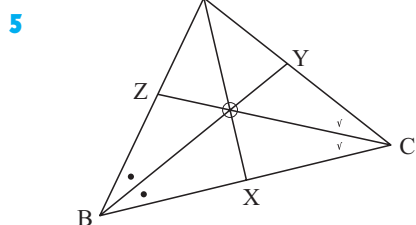
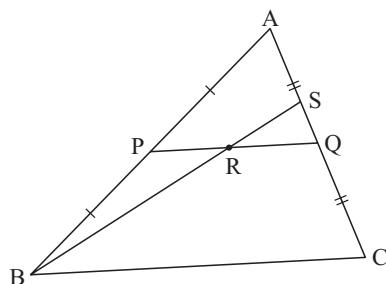
Prove Menelaus' theorem by constructing [AW] parallel to [BX] to meet the transversal at W.

Hint: Look for similar triangles.

- 4 P and Q are the midpoints of sides [AB] and [AC] respectively. R is the midpoint of [PQ].

[BR] produced meets [AC] at S.

Find AS : SC.

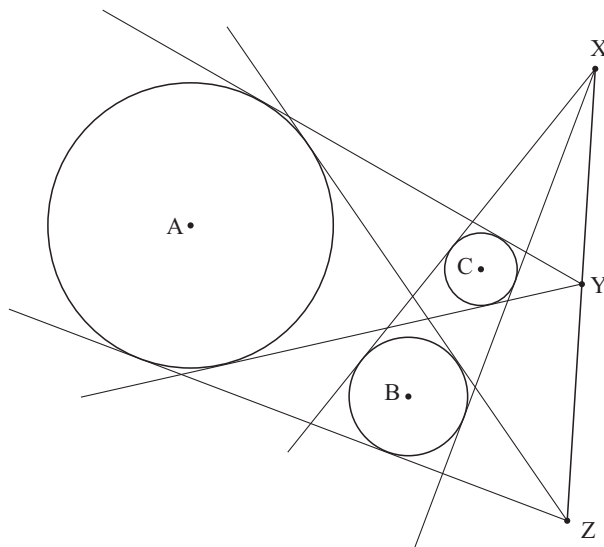


[BY] and [CZ] are internal angle bisectors of angles ABC and ACB. Prove that [AX] bisects angle BAC.

- 6 Common external tangents are drawn for the three pairs of illustrated circles.

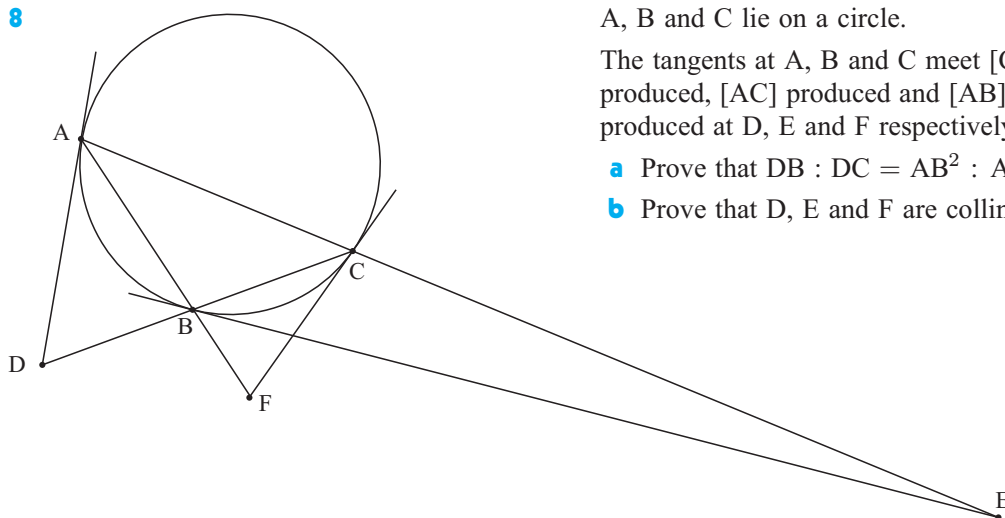
The circles have radii a , b and c units.

Use the converse of Menelaus' theorem to establish that X, Y and Z are collinear.



- 7 Prove Pappus' theorem illustrated at the beginning of this section.

8



A, B and C lie on a circle.

The tangents at A, B and C meet [CB] produced, [AC] produced and [AB] produced at D, E and F respectively.

- a Prove that $DB : DC = AB^2 : AC^2$.
- b Prove that D, E and F are collinear.



EULER'S LINE AND 9-POINT CIRCLE

INVESTIGATION 4

EULER'S LINE AND 9-POINT CIRCLE



For a general triangle ABC, you are to draw on the same figure the medians, the angle bisectors, the altitudes and the perpendicular bisectors of the sides.

We are aware of the concurrence of each type and are interested in the points of concurrence.



What to do:

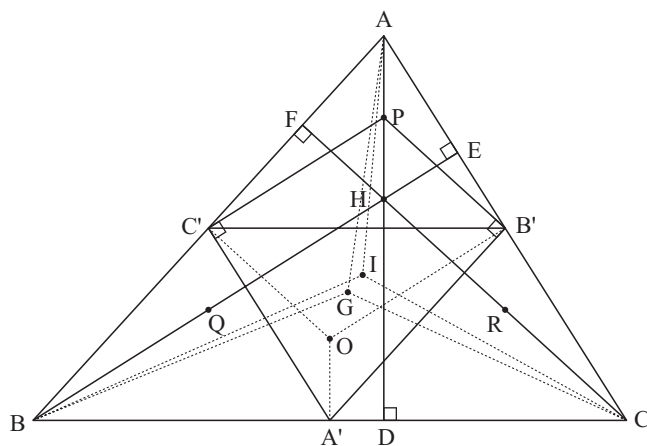
- 1 Label the midpoints of the sides of $\triangle ABC$, A' , B' and C' . (**Note:** A' is opposite A.) Draw the perpendicular bisectors and find O, the point of concurrency. Remove unnecessary construction lines once O has been found.
- 2 Repeat for the medians, locating the centroid G.
- 3 Repeat for the angle bisectors, locating the incentre I.
- 4 Repeat for the altitudes, where D, E and F are the feet of the perpendiculars. Label the orthocentre H.
- 5 Label the midpoints of [AH], [BH] and [CH], P, Q and R respectively.
- 6 Vary the triangle ABC by clicking and dragging any vertex.
- 7 What is the relationship between A' , B' , C' , D, E, F, P, Q and R?
- 8 Once you have done 7 you will be able to define a new point N. There is a relationship between some of O, G, H, N and I. What is it?
- 9 There is one centre which does not fit with the others. Under what conditions does it relate nicely to the other centres?

From the investigation you should have discovered **Euler's line** and **Euler's Nine Point Circle and its centre**.

EULER'S NINE POINT CIRCLE

In the following figure for triangle ABC:

- The midpoints of the sides opposite the vertices are A' , B' , C' .
- The feet of the perpendiculars from the vertices A, B, C are respectively D, E, F.
- The circumcentre of the triangle ABC is O.
- The incentre of the triangle ABC is I.
- The centroid (centre of gravity) of the triangle ABC is G.
- The orthocentre of the triangle ABC is H.
- The midpoints of $[AH]$, $[BH]$ and $[CH]$ are respectively P, Q, R.



EULER'S NINE POINT CIRCLE THEOREM

In triangle ABC, where the sides opposite the vertices have midpoints A' , B' and C' , the feet of the perpendiculars from A, B, C are respectively D, E, F, the circumcentre of triangle ABC is O, the centroid of triangle ABC is G, the orthocentre of triangle ABC is H, and the midpoints of $[AH]$, $[BH]$ and $[CH]$ are respectively P, Q and R, then:

- A' , B' , C' , D, E, F, P, Q and R lie on the same circle.
- The radius of this circle is half that of the circumcircle.
- The centre of the circle is the midpoint of $[OH]$.

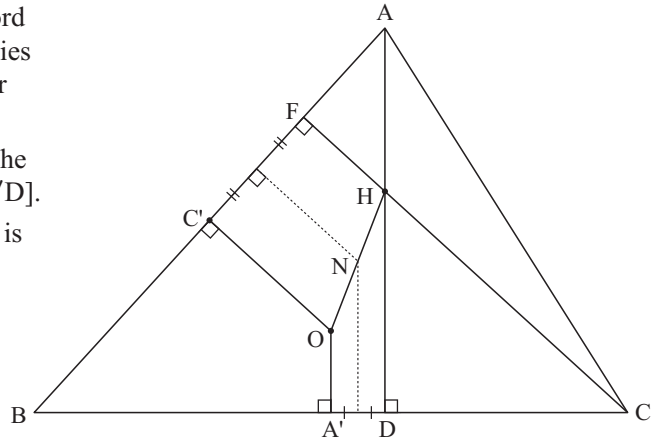
Proof:

- Since $AP = PH$ and $AC' = C'B$, then $[C'P] \parallel [BH]$ {midpoint theorem}
Also, since $BC' = C'A$ and $BA' = A'C$, then $[A'C'] \parallel [AC]$ {midpoint theorem}
But $[BH] \perp [AC]$. Hence, $[C'P] \perp [A'C']$
 $\Rightarrow \angle A'C'P$ is a right angle.
By similar argument, $\angle A'B'P$ is also right angled.
Also $\angle A'DP$ is a right angle {given}
Thus, points C' , B' and D lie on a circle which has diameter $[A'P]$,
i.e., D and P both lie on the circle passing through A' , B' and C' .
Similarly we can show that E and Q, then F and R line on the same circle.
Thus A' , B' , C' , D, E, F, P, Q, R all lie on the same circle.

- The radius of the 9-point circle is half that of the circumcircle of $\triangle ABC$ since $\triangle A'B'C'$ comprises the triangle formed by the lines joining the midpoints of the sides of $\triangle ABC$.
- Notice that as $[C'F]$ is a chord of the circle then its centre lies on the perpendicular bisector of $[C'F]$.

Likewise the centre lies on the perpendicular bisector of $[A'D]$.

So N , the midpoint of $[OH]$ is the centre of the circle.



EXERCISE M

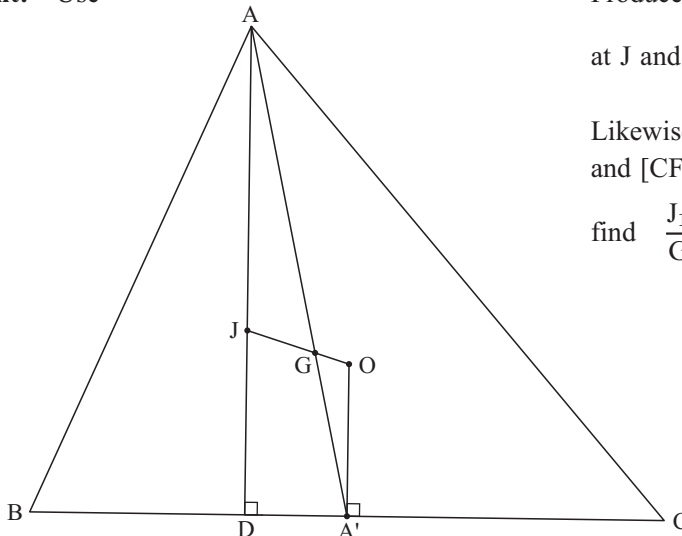
- Find the four triangle centres (O, I, G and H) given that triangle ABC is:
 - isosceles
 - equilateral
 - right angled
 - right angled isosceles.



Click on the icon to help visualise each case.

- If triangle ABC has circumcentre O, centroid G, orthocentre H and nine-point circle centre F, show that OGFH forms a harmonic ratio.
- Prove that O, G and H of any triangle are collinear (where the line passing through them is **Euler's line**).

Hint: Use



Produce $[OG]$ to meet $[AD]$

at J and show that $\frac{JG}{GO} = \frac{2}{1}$.

Likewise, if altitudes $[BE]$

and $[CF]$ meet $[OG]$ at J_1 and J_2

find $\frac{J_1G}{GO}$ and $\frac{J_2G}{GO}$.

EXERCISE C

4 24 m

EXERCISE D

1 a $x = 3.5$ b $x = \frac{\sqrt{105} - 5}{2}$ c $x = \sqrt{41} - 4$

2 a $DX = 4.8$ cm b $CD = 7$ cm

c $CX = \frac{9 \pm \sqrt{21}}{2}$ cm d $r = \sqrt{41}$ cm

e $AB = 3\sqrt{6}$ cm

3 a $CD = 5$ cm b $AB = 2\sqrt{11}$ cm

4 a $BX = 9$ cm b $XT = \sqrt{10}$ cm c $OX = 11$ cm

5 a 2293 km

7 Show $AX \times BX = CX \times DX$

8 Show $AX \times BX = CX^2$

9 $AP = 3$ cm

EXERCISE E.1

1 a centre (2, 3), radius 2 units

b centre (0, -3), radius 3 units

c centre (2, 0), radius $\sqrt{7}$ units

2 a $(x-2)^2 + (y-3)^2 = 25$ b $(x+2)^2 + (y-4)^2 = 1$

c $(x-4)^2 + (y+1)^2 = 3$ d $(x+3)^2 + (y+1)^2 = 11$

3 a $(x-3)^2 + (y+2)^2 = 4$ b $(x+4)^2 + (y-3)^2 = 16$

c $(x-5)^2 + (y-3)^2 = 17$ d $(x-2)^2 + (y-2)^2 = 17$

e $(x+3)^2 + (y-2)^2 = 7$

4 a a circle, centre (-2, 7), radius $\sqrt{5}$ units

b the point (-2, 7)

c nothing, the equation is impossible in the real system

6 a on the circle b inside the circle c outside the circle

d outside the circle

7 a $k = 5$ or -1 b $k = -2 \pm \sqrt{11}$ c $k = 3$ or -1

EXERCISE E.2

1 a $(-3, 1)$, $\sqrt{13}$ units b $(3, 0)$, $\sqrt{11}$ units

c $(0, -2)$, $\sqrt{5}$ units d $(-2, 4)$, $\sqrt{17}$ units

e $(2, 3)$, 4 units f $(4, 0)$, 4 units

2 a $k = 36$ b $k = -2$ c $k < 5$

3 a a circle, centre $(\frac{11}{2}, 0)$, radius $\frac{3}{2}$ units

b a circle, centre $(\frac{1}{2}, 0)$, radius $\frac{3}{2}$ units

c the vertical line $x = 3$

4 a P lies on a circle with equation $x^2 + y^2 - \frac{44}{3}x + \frac{140}{3} = 0$.

Its centre is $(\frac{22}{3}, 0)$ and radius is $\frac{8}{3}$ units.

EXERCISE E.3

4 If the circles intersect at A and B then M lies on the line through A and B, external to [AB].

5 M lies on the common tangent through the point of contact K.

8 a 0, M lies on C b 23, M lies outside C

c -3, M lies inside C

9 a $(0, 2)$ and $(2, 0)$ b $y = -x + 2$, $0 \leq x \leq 2$

c $y = -x + 2$

d $y = -x + 2$, the equation of the radical axis

10 a $(1, -3)$ and $(3, 3)$ b $y = 3x - 6$, $1 \leq x \leq 3$

c $y = 3x - 6$

d $y = 3x - 6$, the equation of the radical axis

11 a $x + 3y = 3$ Yes, at $(3, 0)$ and $(\frac{18}{5}, -\frac{1}{5})$.

b $4x - 2y = 7$

c $y = 4\frac{1}{2}$, $4x + 12y = 39$, $(-\frac{15}{4}, \frac{9}{2})$

EXERCISE G.2

1 ≈ 8.85 cm 2 ≈ 15.9 cm 3 $\sqrt{20}$ cm, $\sqrt{8}$ cm, $\frac{10}{\sqrt{3}}$ cm

4 $4\sqrt{10}$ cm

EXERCISE G.3

2 $\sqrt{47}$ cm 3 ≈ 11.5 cm

EXERCISE G.4

1 15.5 cm 2 $\frac{30}{7}$ cm 3 a No

EXERCISE H.1

1 a 13 cm b $\frac{25}{13}$ cm c $\frac{49}{13}$ cm

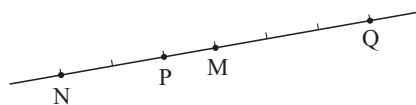
EXERCISE H.2

1 $\triangle ABC : \triangle BCD = 5 : 3$

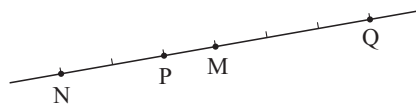
2 a 16 : 49 b 16 : 23 c $\frac{234}{23}$ cm² 3 $\frac{1}{2}$ 4 $\frac{5}{24}$

EXERCISE I

1 a

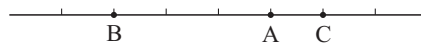


b

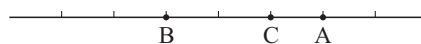
**EXERCISE J**

1 a 4 : 3 b 3 : 4 c -7 : 3 d -3 : 7 e -4 : 7
f -7 : 4

2 a



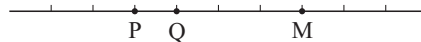
b



c



d



3 2 : 1

EXERCISE K

1 a 1 : 2 b 3 : 1 c 1 : 3

2 $\frac{AP}{PB} \times \frac{BQ}{QC} \times \frac{CR}{RA} = \frac{2}{1} \times \frac{3}{2} \times \frac{1}{6} = 1$ etc.

4 1 : 4

EXERCISE L

1 -10 : 9 2 2 : 1